http://www.Lohmander.com/PlantCostMin20100112.pdf

# Plantation Cost Minimization

*Peter Lohmander 2010-01-12*

$$\min_{x,y} C = px + ry$$

*s.t.*

$$ax - b\frac{x^2}{y} = z$$

$$(Az = q)$$

$$a > 0; \quad b > 0; \quad p > 0; \quad r > 0; \quad z > 0$$

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$$x^* = \frac{z}{a}\left(1 + \sqrt{\frac{1}{\frac{ap}{br} + 1}}\right)$$

$$y^* = \frac{bx^2}{ax - z}$$

$$C^* = px^* + ry^*$$

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$$q = Az(x, y)$$

$$z = ax - b\frac{x^2}{y}$$

$$(z - ax)y = -bx^2$$

$$(ax - z)y = bx^2$$

$$y = \frac{bx^2}{ax - z}$$

$$C = C(x, y) = C(x) = px + ry(x)$$

$$C(x) = px + r\left(\frac{bx^2}{ax - z}\right)$$

$$C(x) = px + rb\left(\frac{x^2}{ax - z}\right)$$

$$\frac{\partial C}{\partial x} = p + rb\left(\frac{2x(ax-z)-ax^2}{(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + rb\left(\frac{2ax^2-2xz-ax^2}{(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + rb\left(\frac{ax^2-2xz}{(ax-z)^2}\right)$$

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$$(ax-z)^2 = a^2x^2 - 2axz + z^2$$

$$\frac{1}{a}(ax-z)^2 = ax^2 - 2xz + \frac{z^2}{a}$$

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$$\frac{\partial C}{\partial x} = p + rb\left(\frac{\frac{1}{a}(ax-z)^2 - \frac{z^2}{a}}{(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + rb\left(\frac{1}{a} - \frac{z^2}{a(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + \frac{rb}{a} - \frac{rb}{a}\left(\frac{z^2}{(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + \frac{rb}{a} - \frac{rb}{a}\left(\frac{z^2}{(ax-z)^2}\right) = 0$$

$$\frac{\partial C}{\partial x} = (ax-z)^2\left(p + \frac{rb}{a}\right) - \frac{rb}{a}z^2 = 0$$

$$\frac{\partial C}{\partial x} = (a^2x^2 - 2azx + z^2)\left(p + \frac{rb}{a}\right) - \frac{rb}{a}z^2 = 0$$

$$\frac{\partial C}{\partial x} = a^2x^2 - 2azx + z^2 - \frac{rbz^2}{a\left(p + \frac{rb}{a}\right)} = 0$$

$$\frac{\partial C}{\partial x} = x^2 - \frac{2z}{a}x + \frac{z^2}{a^2} - \frac{rbz^2}{a^3\left(p + \frac{rb}{a}\right)} = 0$$

$$\frac{\partial C}{\partial x} = x^2 - \frac{2z}{a}x + z^2\left(\frac{1}{a^2} - \frac{rb}{a^3\left(p + \frac{rb}{a}\right)}\right) = 0$$

$$x = -\left(\frac{-\frac{2z}{a}}{2}\right) + (\vee -)\sqrt{\left(\frac{-\frac{2z}{a}}{2}\right)^2 - z^2\left(\frac{1}{a^2} - \frac{rb}{a^3\left(p + \frac{rb}{a}\right)}\right)}$$

$$x = \frac{z}{a} + (\vee -)\sqrt{\frac{z^2}{a^2} - \frac{z^2}{a^2} + \frac{rbz^2}{a^3\left(p + \frac{rb}{a}\right)}}$$

$$x = \frac{z}{a} + (\vee -)z\sqrt{\frac{rb}{a^3\left(p + \frac{rb}{a}\right)}}$$

$$x = \frac{z}{a} + (\vee -)z\sqrt{\frac{rb}{a^3 p + a^2 rb}}$$

$$x = \frac{z}{a}\left(1 + (\vee -)\sqrt{\frac{rb}{ap + rb}}\right)$$

$$x = \frac{z}{a}\left(1 + (\vee -)\sqrt{\frac{1}{\frac{ap}{br} + 1}}\right)$$

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$$\left[y = \frac{bx^2}{ax - z} > 0\right] \Rightarrow \left[x > \frac{z}{a}\right] \Rightarrow x = \frac{z}{a}\left(1 + \sqrt{\frac{1}{\frac{ap}{br} + 1}}\right)$$

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$$\frac{\partial C}{\partial x} = p + \frac{rb}{a} - \frac{rb}{a}\left(\frac{z^2}{(ax-z)^2}\right)$$

$$\frac{\partial C}{\partial x} = p + \frac{rb}{a} - \frac{rbz^2}{a}(ax-z)^{-2}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{2rbz^2}{a}(ax-z)^{-3}a$$

$$\frac{\partial^2 C}{\partial x^2} = 2rbz^2(ax-z)^{-3}$$

$$\left[x > \frac{z}{a}\right] \Rightarrow \left(\frac{\partial^2 C}{\partial x^2} > 0\right) \Rightarrow \text{"min"}$$

$$\left[x < \frac{z}{a}\right] \Rightarrow \left(\frac{\partial^2 C}{\partial x^2} < 0\right) \Rightarrow \text{"max"}$$

$$\text{"min"} \Rightarrow \left[\frac{\partial^2 C}{\partial x^2} > 0\right] \Rightarrow \left(x > \frac{z}{a}\right) \Rightarrow x = \frac{z}{a}\left(1 + \sqrt{\frac{1}{\frac{ap}{br} + 1}}\right)$$