National Conference on the Caspian forests of Iran
"Past, Current, Future"
University of Guilan, Rasht, Iran, April 26-27, 2017

A general dynamic function for the basal area of individual trees derived from a production theoretically motivated autonomous differential equation

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Abstract

A general dynamic function for the basal area of individual trees has been derived from a production theoretically motivated autonomous differential equation. The differential equation is $\frac{dx}{dt} = a\sqrt{x}\left(1-cx\right)$, $a>0, c>0, 0<x<c^{-1}$ and the general dynamic function is: $x(t)=\frac{\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}+1\right)^2}{c\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}-1\right)^2}$.

Keywords: Dynamic function, differential equation, basal area, forest growth

Introduction

The mathematical methods utilized in this paper are well presented by Braun (1983) and Simmons (1972). Now, fundamental biological production theory will suggest a relevant differential equation. We start with a small tree. Consider a stem segment, of height H, of the tree. The stem segment is cylindrical with diameter D_1. The leaves cover a cylinder with diameter D_2. $D_2 = \gamma D_1, \gamma > 1$. The sun light reaches the tree from the side. V is the volume of the stem segment. x is the basal area. $x = (\pi/4)D_1^2$. $V = Hx$. Volume increment is proportional to the photo synthesis level, P, which in turn is proportional to the sun light projection area on the leaves, A. $A = HD_2$. We may conclude that:

$\frac{dV}{dt} = H\frac{dx}{dt} \propto P \propto A \propto D_2 \propto D_1 \propto \sqrt{x}$. Hence, $\frac{dx}{dt} \propto \sqrt{x}$ or $\frac{dx}{dt} = a\sqrt{x}, a > 0$. As the size of the tree increases, the production efficiency declines. Furthermore, the value of γ is often lower for large trees than for small trees. A relevant function considering this is: $\frac{dx}{dt} = a\sqrt{x}(1-cx) \quad a > 0, c > 0, 0 < x < c^{-1}$.

Mathematical model development and analysis

$$\frac{dx}{dt} = ax^{0.5} - bx^{1.5} \qquad a > 0, b > 0, 0 < x < \frac{a}{b}$$

$$\frac{dx}{dt} = a\sqrt{x}(1-cx) \quad , \quad c = \frac{b}{a} > 0, 0 < x < c^{-1}$$

$\frac{1}{\sqrt{x}(1-cx)}dx = a\,dt$. Integration gives $\int \frac{1}{\sqrt{x}(1-cx)}dx = \int a\,dt + k_0$.

$\frac{\ln\left(\sqrt{c}\sqrt{x}+1\right) - \ln\left(\sqrt{c}\sqrt{x}-1\right)}{\sqrt{c}} = at + k_0$. Let us investigate the left hand side, called Z.

$$\frac{dZ}{dx} = \frac{1}{\sqrt{c}\left(\sqrt{c}\sqrt{x}+1\right)}\left(\frac{\sqrt{c}}{2\sqrt{x}}\right) - \frac{1}{\sqrt{c}\left(\sqrt{c}\sqrt{x}-1\right)}\left(\frac{\sqrt{c}}{2\sqrt{x}}\right)$$

$\frac{dZ}{dx} = \frac{1}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}+1\right)} - \frac{1}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}-1\right)}$. Then, $\frac{dZ}{dx} = \frac{\left(\sqrt{c}\sqrt{x}-1\right)-\left(\sqrt{c}\sqrt{x}+1\right)}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}+1\right)\left(\sqrt{c}\sqrt{x}-1\right)}$

$\frac{dZ}{dx} = \frac{1}{\sqrt{x}(1-cx)}$. This confirmes that the integration was correct.

$\frac{\ln\left(\frac{\sqrt{c}\sqrt{x}+1}{\sqrt{c}\sqrt{x}-1}\right)}{\sqrt{c}} = at + k_0$ which leads to $\ln\left(\frac{\sqrt{c}\sqrt{x}+1}{\sqrt{c}\sqrt{x}-1}\right) = \sqrt{c}at + k_1 \quad k_1 = \sqrt{c}k_0$.

Let $y = \sqrt{x}, g = \sqrt{c}, h = ga$. Then, $\ln\left(\frac{gy+1}{gy-1}\right) = ht + k_1$.

Let $K = e^{k_1}$. We get the simplified expression: $\frac{gy+1}{gy-1} = Ke^{ht}$ which can be transformed to: $gy+1 = Ke^{ht}(gy-1)$ or $g(1-Ke^{ht})y = -1-Ke^{ht}$ and $y = \frac{-1-Ke^{ht}}{g(1-Ke^{ht})}$. $y = \sqrt{x} = \frac{Ke^{ht}+1}{g(Ke^{ht}-1)}$ which gives the desired equation $x(t) = \frac{(Ke^{ht}+1)^2}{c(Ke^{ht}-1)^2}$.

Let us determine K. We utilize the initial condition: $x_0 = x(0)$. $\sqrt{x_0} = \frac{(Ke^0+1)}{\sqrt{c}(Ke^0-1)}$ which leads to $\sqrt{x_0}\sqrt{c}(K-1) = K+1$. Hence, $(\sqrt{x_0}\sqrt{c}-1)K = \sqrt{x_0}\sqrt{c}+1$ and finally

$K = \frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}$. Now, we know how to determine K. Later, the sign and magnitude of K will be needed in the analysis. Do we know the sign of K?

$(\sqrt{x_0} > 0 \wedge \sqrt{c} > 0) \Rightarrow (\sqrt{x}\sqrt{c}+1 > 0)$. Let us investigate the sign of $\sqrt{x_0}\sqrt{c}-1$. We assume that the value of x_0 makes sure that the increment is strictly positive.

$\frac{dx}{dt} = a\sqrt{x}(1-cx)$. Then, we know that:

$$1-cx_0 > 0 \quad \Rightarrow \quad cx_0 < 1 \quad \Rightarrow \quad \sqrt{c}\sqrt{x_0} < \sqrt{1} \quad \Rightarrow \quad \sqrt{x_0}\sqrt{c}-1 < 0$$

As a result, we know that $K < 0$. Do we know the something about $|K|$?

$$K = \frac{\phi+1}{\phi-1} \quad 0 < \phi = \sqrt{x_0}\sqrt{c} < 1.\ K(\phi = \varepsilon) = \frac{\varepsilon+1}{\varepsilon-1} \quad \Rightarrow \quad \lim_{\varepsilon \to 0} K(\phi = \varepsilon) = -1.$$

$$\frac{dK}{d\phi} = \frac{(\phi-1)-(\phi+1)}{(\phi-1)^2} \quad \Rightarrow \quad \frac{dK}{d\phi} = \frac{-2}{(\phi-1)^2} < 0.$$

With this information, we know that $K < -1$. Now, we may determine $x(t)$ as an explicit function of x_0 and the parameters.

$$x(t)=\frac{\left(Ke^{ht}+1\right)^2}{c\left(Ke^{ht}-1\right)^2} \quad \wedge \quad K=\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1} \quad \wedge \quad h=a\sqrt{c} \quad \Rightarrow$$

$$x(t)=\frac{\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}+1\right)^2}{c\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}-1\right)^2}$$

Now, the dynamic properties of $x(t)=\frac{\left(Ke^{ht}+1\right)^2}{c\left(Ke^{ht}-1\right)^2}$ will be determined.

$$\frac{dx}{dt}=\left(\frac{2Khc}{c^2\left(Ke^{ht}-1\right)^4}\right)\left(\left(Ke^{ht}+1\right)\left(Ke^{ht}-1\right)^2-\left(Ke^{ht}+1\right)^2\left(Ke^{ht}-1\right)\right)$$

$$\frac{dx}{dt}=\left(\frac{2Khc\left(Ke^{ht}+1\right)\left(Ke^{ht}-1\right)}{c^2\left(Ke^{ht}-1\right)^4}\right)\left(\left(Ke^{ht}-1\right)-\left(Ke^{ht}+1\right)\right)$$

$$\frac{dx}{dt}=\left(\frac{-4Khc\left(\left(Ke^{ht}\right)^2-1\right)}{c^2\left(Ke^{ht}-1\right)^4}\right)$$

We already know that $K<-1$. Hence, $\left(\left(Ke^{ht}\right)^2-1\right)>0$. As a result, we find that $\frac{dx}{dt}>0$.

$$\lim_{\substack{t\to\infty\\ h>0\\ K<-1}} x(t)=\frac{\left(\frac{d\left(Ke^{ht}+1\right)^2}{dt}\right)}{\left(\frac{dc\left(Ke^{ht}-1\right)^2}{dt}\right)}=\frac{2\left(Ke^{ht}+1\right)hK}{2c\left(Ke^{ht}-1\right)hK}=\frac{\left(1+\frac{1}{Ke^{ht}}\right)}{c\left(1-\frac{1}{Ke^{ht}}\right)}=\frac{1}{c}$$

Hence, we know that, as $t\to\infty$, $x(t)$ monotonically converges to c^{-1}.

Results

A general dynamic function for the basal area of individual trees has been derived from a production theoretically motivated autonomous differential equation. The dynamic properties have been determined and monotone convergence has been proved.

Discussion

The production theoretically motivated autonomous differential equation has recently been tested with regression analysis, using two sets of forest data from the Caspian forests. Detailed reports will soon be available. The parameters obtained the expected signs and the t-values indicated very high precision. The adjusted R2-values, the very high F-values and the lack of unexplained trends in the residual graphs, indicated a relevant model. The functional form has been found to work very well with empirical data from Iran and hopefully it will be possible to test it also on forest data from other regions of the world. It is possible to use the model also under the influence of some forms of size dependent competition, via adjustments of the parameters (a,b). Further developements in this direction should be expected. Presently, different forest research projects with a strong focus on empirical investigations are active, in Sweden and in Iran. Hopefully, the derived analytical model will be useful in these projects.

Conclusions

Differential equations can be very useful tools in forest management optimization. It is however important that they have theoretical and empirical support. Future efforts should be directed towards applications of the suggested and derived function within forest management problems in several regions of the world.

Acknowledgments

The author is grateful to Associate Professor Soleiman Mohammadi Limaei who arranged several forest excursions and meetings in Iran. Furthermore, he arranged meetings and cooperation with seriously interested PhD students with detailed forest data from the Caspian forests. Professor Johan Bergh, at Linnaeus University in Sweden, coordinates a very interesting project on optimal continuous cover forestry and creates a very positive research environment. Thank you!

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