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Optimal Resource Control in Continuous Time without Hamiltonian Functions

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The optimal intertemporal control (extraction or harvest) problem which arises in resource management has earlier been analysed in continuous time by the use of Hamiltonian functions and classical optimal control theory. This paper presents a convenient alternative to that approach. The constrained optimization problem is formulated in discrete time. The Lagrange function is constructed and the first order optimum conditions are derived. From these and the dual variable, the optimal control path is determined. A similar method can be used also in continuous time since the periods may represent time intervals that approach zero. The dual variable is determined via the resource constraint and integration over time. Thanks to the use of classical constrained optimization, the second order maximum conditions are easily investigated and it is often simple to verify that the solution is a unique maximum. One application to forest harvesting and one to oil field extraction are analysed. A computer program based on the analytical results is included and used in the applications.

1. Introduction

1.1. General Introduction

The optimal intertemporal control (harvest or extraction) problem which arises in resource management has earlier been analysed in continuous time by the use of Hamiltonian functions and classical optimal control theory. This paper presents a convenient alternative to that approach. Furthermore, two typical applications to resource management will be discussed.

The problem under investigation is the following: You own a natural resource and the optimal extraction (harvest) level should be determined as a function of time. Depending on the situation, different methods are possible. Some general properties of the problem must always be taken into consideration.

1.2. Basic Considerations

The first question is if the resource grows (renewable) or not (nonrenewable or exhaustible). Secondly, we want to know the nature of the profit function: Do we affect the price level through our supply or can we regard the price as an exogenous parameter ? Do we buy the input factors on perfect markets or do we affect the factor prices through our activities ? Is the marginal cost affected by the extraction (harvest) level or/and the size of the resource stock? In most papers in the field, rather strong assumptions concerning these questions are often made without much discussion. A nice presentation of many central problems and the economic implications can be found in CLARK [2] and in JOHANSSON und LÖFGREN [6].

1.3. Deterministic or Stochastic Methods

One very important question is the following: Is it possible to predict the price and/or growth in the future? Is the assumption of a deterministic environment really relevant? If this is not the case, it is in most situations better to use the latest information in the decision process. The final decisions should be made as late as possible and no detailed long term planning should be performed. This is called adaptive optimization. The mathematical theory of stochastic optimal control can be found in FLEMING and RISHEL [4]. An introduction to some of the economic implications are presented by HEY [5]. Adaptive optimization applications to resource management are discussed by NORSTROM [10], RISVAND [11] and LOHMANDER [7, 8 and 9] among others. The reader should be aware that adaptive optimization generally leads to much better results than deterministic optimization when the price and growth processes are in fact not deterministic (and the deterministic method is used as an approximation).

However, the deterministic control approaches (discussed in this paper and elsewhere), may be applied in situations when good forecasts are possible. Furthermore, they serve as convenient analytical tools in the discussion of intertemporal resource problems. But, since the adaptive models generally are more relevant, one should not hesitate to consult the more difficult adaptive optimization approach in important cases.

1.4. First and Second Order Properties

This paper presents an analytically convenient and powerful method of deterministic optimal control in continuous (and discrete) time and applications to intertemporal resource extraction (harvesting). It is based on classical constrained optimization. Hence, the solution has important and well known analytical properties. For instance, it is easy to investigate the second order maximum conditions. In many cases, the derived solution is easily shown to be a unique intertemporal maximum.

In resource management applications of classical optimal control theory with Hamiltonian functions, the necessary (first order) conditions are used to derive the optimal time path of the harvest. It is frequently difficult to show that the derived solution really is a maximum (or even a unique maximum). The existence of optimal controls in resource problems is discussed by CLARK [2, p. 168] and the general control problem by BERKOVITZ [1, Corollary 5.1., p. 67]. Clark, following Berkovitz, states that: "*If the integrand* $g(x, t, u)$ (corresponding to the profit $\pi(h(t))$ in this paper) *is concave downward as a function of* u (corresponding to the extraction or harvest level $h(t)$ in this paper), *then an optimal control* (in the usual sence) *exists*."

Note that this is not always true, at least not if the resource grows in the more general manner, suggested in this paper. Then, in order to show that the control derived from the first order optimum conditions really is a (unique) maximum, we must show that the objective function Π, the time integral of π, is (strictly) concave in $h(t)$ for all t. This is generally much more difficult, since the properties of the growth function enter the derivations. It is quite possible that $\pi(t)$ is concave in $h(t)$ *and* that Π is strictly convex in $h(t)$ in the same problem! $h(t)$ will generally affect the growth until the following period $t + 1$ through a nonlinear, concave *or* convex, relationship. This, in turn, will affect Π through $h(t+1)$ and hence $\pi(h(t+1))$. Thus, Π may be strictly convex in $h(t)$, depending on the growth function, even if π is strictly concave in $h(t)$.

These questions are discussed when the growth is governed by a controlled stochastic Markov diffusion process by LOHMANDER [9] and solutions are derived in particular cases. Some references on convexity/concavity questions and intertemporal optimization methods applied to forest harvesting are presented by VALSTA [12].

1.5. Other Improvements

One novelty of this paper is that the resource is introduced in the shape of a general intertemporal resource constraint. This is a strong generalization of classical approaches where the resource dynamics is described by a controlled Markov diffusion process. The Markov specification is no longer necessary (even though the author admits that Markov models may be relevant in many cases).

An important feature of the new approach is that modifications easily are introduced. New restrictions and activities can be included and the relevance of the method may hence increase in particular normative cases.

2. Analysis

The structure of the analysis is the following. The optimization problem is defined in the mathematical appendix. There, the connections between discrete and continuous time are explicitly discussed. The optimal control path and the marginal resource value are derived as explicit functions of the parameters. In the calculations, the first order optimum conditions are used. The second order conditions show that the derived solution is a unique maximum.

In the numerical appendix, a computer program is included, which is based on the analytically derived results. Two natural resource management applications are specified in Section 3; CASE 1 is a nonrenewable resource problem and CASE 2 is a forest harvesting problem. Graphs derived from the computer program are used in Section 5 in the discussion of the optimal solutions and the sensitivity to the parameters.

3. Two Important Applications

3.1. The Case of the Nonrenewable Resource (Case 1)

We consider the problem of deciding the optimal extraction of a resource over time. The stock of the resource at time 0 is Q_0. The resource should be extracted between time 0 and T. There is no growth. Hence, the time integral of the integrand $h(t)$, the harvest level, from 0 to T should be equal to Q_0. This is illustrated in Fig. 1. Of course, the optimal choice of $h(t)$ is dependent on all parameters in the problem. There is an infinite number of possible functions $h(t)$. Mathematically, the problem is stated as:

$$\text{Max}\ \Pi = \int_0^T e^{-rt}\,\pi(h(t))\,dt$$

$$s.t.\ \int_0^T h(t)\,dt \overset{(\leq)}{=} Q_0$$

where Π is the present value of all extraction, r is the rate of discount in the capital market, $h(t)$ is the extraction (harvest) level and $\pi(.)$ denotes the profit gained at time t.

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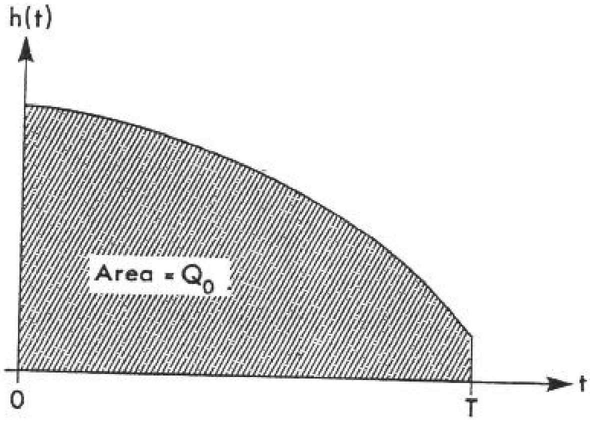

Fig. 1. The harvest $h(t)$ as a function of time. CASE 1, an exhaustible resource

3.2. The Case of Forest Harvesting (Case 2)

This case may be considered as a generalization of CASE 1. We assume that the growth rate of the stock which has not yet been harvested is constant.

This may be a correct description of a forest where the tree age is almost the same in all parts of the area and other conditions are held constant. The relative growth does not change very much during the period of consideration. All harvests are performed as clear fellings. Clearly, the growth in one part of the area is not affected by the size of the stock in other parts of the area. (Stock dependence is generally assumed in the growth process in applied control theory. That may be relevant in case harvesting is performed

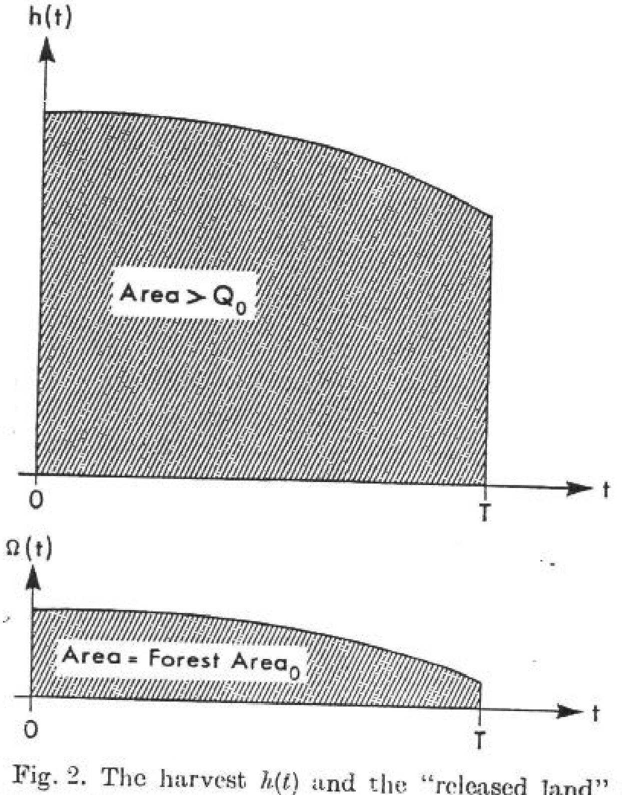

Fig. 2. The harvest $h(t)$ and the "released land" $\Omega(t)$ as function of time. CASE 2, a renewable resource

through thinnings.) CASE 2 is different from CASE 1 also because the land typically has a value in the production of new tree generations. We may consider a value of the land occupied by the resource in CASE 1 too. However, that value must then be based on something else than new production of the "nonrenewable" resource in question. Since the forest (stand density) grows, the land area "released" per harvested unit (cubic metre) is a decreasing function of time. Mathematically, CASE 2 is formulated as:

$$\text{Max}\, \Pi = \int_0^T e^{-rt}\, \pi(h(t), \Omega(t))\, dt, \quad \Omega(t) = g\alpha^t h(t)$$

$$s.t. \int_0^T \alpha^t h(t)\, dt \overset{(\leq)}{=} Q_0, \quad 0 < \alpha < 1.$$

The definitions are the same as in CASE 1. $\Omega(t)$ is the land area released after harvest and α is a parameter (consistent with a constant growth rate). Clearly, the simple growth assumption could be generalized. This will however not be made here. The fundamental properties of (and differences between) CASE 1 and CASE 2 are illustrated in the Figs. 1 and 2 respectively.

4. The Properties of the Profit Function

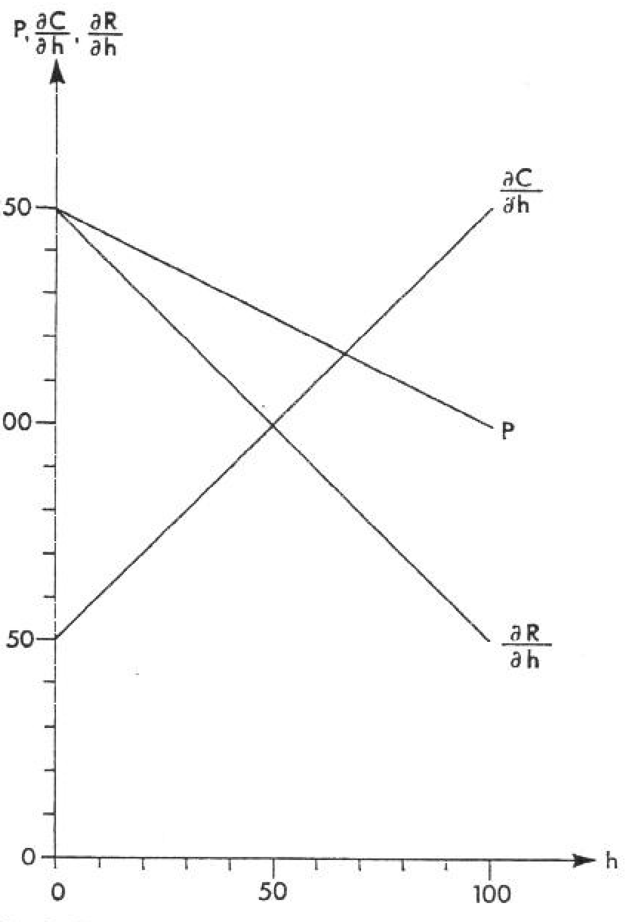

Fig. 3. Price, marginal cost and marginal revenue. $P(h) = 150 - .5h$, $C(h) = F + (50 + .5h)\,h$, $R(h) = P(h)\,h = 150h - .5h^2$

In order to make the discussion and the calculations more specific, we assume that the profit function $\pi(h)$ is derived from the price and cost conditions found in Fig. 3. There, the "base" parameters to be discussed and deviated from in the numerical analysis are shown. The price of the product, P, is a linear function of the supply from the enterprise, h. The marginal cost $C'(h)$ is a linear function of the harvest (extraction) volume h. In most cases, $P'(h) < 0$ and $C'(h) > 0$. $R(h)$ denotes revenue, and is equal to $P(h)\,h$.
Of course, the profit $\pi(h)$ becomes a quadratic function of h.

$$\pi(h) = P(h)\,h - C(h).$$

If we make use of the numerical details, we find that (F denotes the fix cost):

$$\pi(h) = (150 - .5h)\,h - F - (50 + .5h)\,h$$
$$\pi(h) = 100h - h^2 - F.$$

The profit is illustrated as a function of the harvest level and the fix cost in Fig. 4. Clearly, the optimal value (if we only consider this particular period) of π is 2500. This is obtained also through the first order optimum condition (one period optimum!):

$$\pi'(h) = 100 - 2h = 0.$$

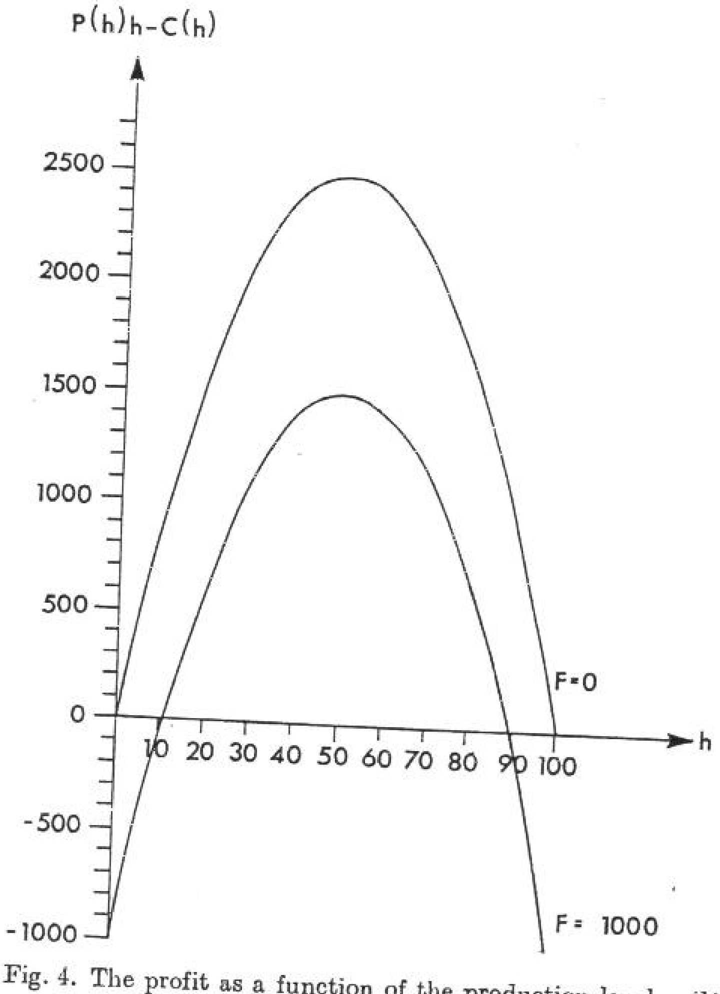

Fig. 4. The profit as a function of the production level. $\pi(h) = P(h)\,h - C(h) = -F + 100\,h - h^2$

Obviously, the harvest level must be 50 units when π reaches the maximum. This is true in the graph. It is easily found that $\pi(h)$ is strictly concave in h!

$$\pi''(h) = -2 < 0.$$

This is also consistent with the graph and makes sure that there is only one unique maximum in the one period problem.

5. The Optimal Solutions

In this section, the optimal extraction (or harvesting) strategy is investigated through optimization via the computer program included in the numerical appendix. The computer program is based on the analytical derivations to be found in the mathematical appendix. (In the graphs and the computer program, the upper case letters A, B and G replace the lower case letters a, b and g, used in the derivations, for technical reasons.)

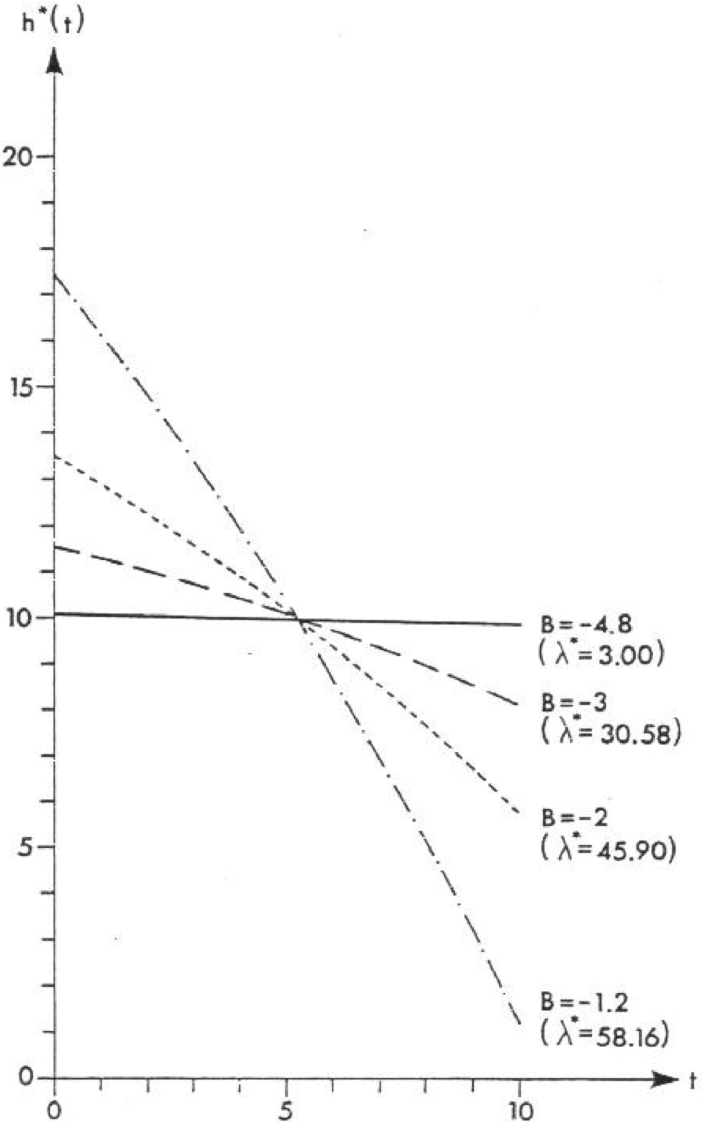

Fig. 5. The optimal harvest path $h^*(t)$ as a function of the concavity of the profit function. CASE 1, an exhaustible resource. The growth rate $\rightarrow 0$ and the land has no value. $\pi(h) = -F + 100h + Bh^2$, $\alpha \rightarrow 1 (\alpha = .9999)$, $\beta = .95$, $Q_0 = 100$, $G = 0$, $T = 10$

Fig. 5 shows the optimal harvest level in CASE 1 as a function of the concavity of the profit function, $\pi(h)$. Since CASE 1 represents a situation without growth, the initial resource Q_0 is identical to the total extraction over the time interval. (If the growth rate is zero, $\alpha = 1$. However, in order to avoid division by zero, we let $\alpha \to 1$ ($\alpha = .9999$).) This can easily be checked in the graph $\left(\int_0^T h(t)\,dt = Q_0 = 100\right)$. If π is not very concave, ($B = -1.2$), it is optimal to extract much in the beginning of the period and little in the end, since the rate of interest is positive ($\beta = e^{-r} < 1$). In fact, if π would have been linear in h, then it would have been optimal to extract all of the resource at $t = 0$. When π becomes more concave, it is optimal to distribute the extraction more evenly over the years. Deviations from the "ideal" (optimal with respect to the period in question only) extraction level become more costly. In the most concave case ($B = -4.8$), the optimal extraction level is almost a constant ($=10$). Fig. 5 also includes information concerning the marginal resource value, the optimal dual variable λ^*. It is found that λ^* is an increasing function of B(because when B increases, then the marginal profit of harvesting increases in each period).

Fig. 6 (CASE 1) shows the optimal intertemporal price policy (consistent with the extraction path demonstrated in Fig. 5) as a function of the profit function concavity.

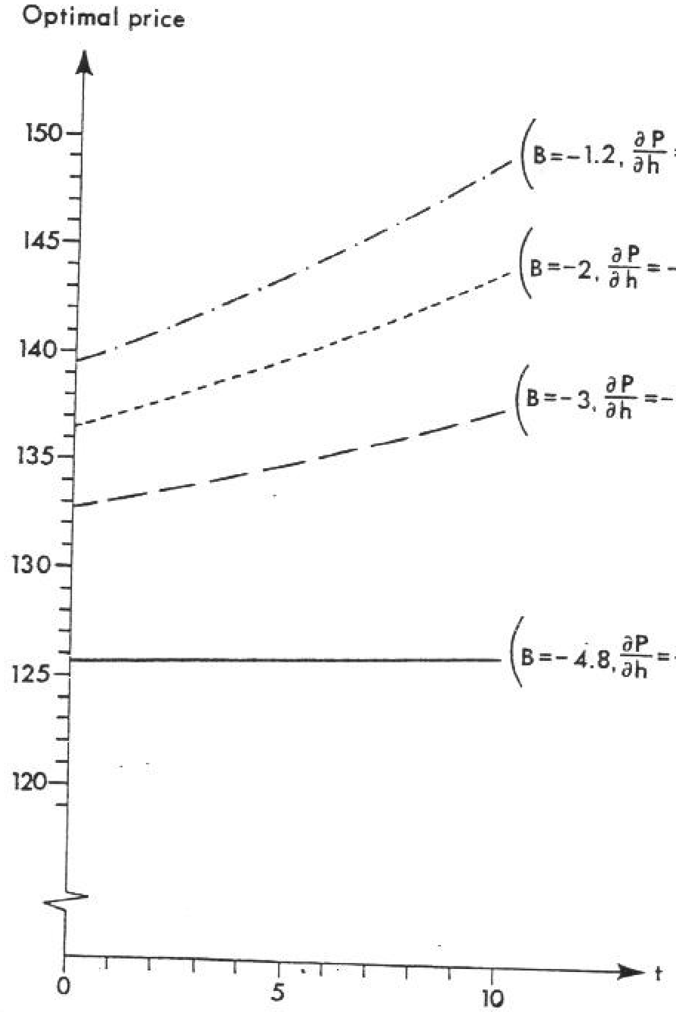

Fig. 6. The optimal prices consistent with the production $h^*(t)$ shown in Fig. 5. CASE 1, an exhaustible resource. Compare Fig. 5

In all cases, the price increases over the years because of a decreasing extraction level. When the profit function π is almost linear ($B = -1.2$), the extraction level (and hence the optimal price) changes very much during the period. When π is very concave ($B = -4.8$), the extraction level and the optimal price are almost constant.

In Fig. 7 (CASE 2), the optimal harvest path is derived as a function of the rate of interest ($\beta = e^{-r}$). As expected, we find that the present harvest level increases and the future harvest level decreases if the rate of interest increases (β decreases). Since the profit function π is strictly concave in h, it is not optimal to harvest all of the resource in one period only, which would be the case in a linear world. If the land value would have been zero, then the optimal harvest level would have been a constant (when the growth rate is equal to the rate of interest ($\alpha = \beta = .95$)). The *qualitative* result can be found in Appendix M. 2, Remark 1. However, since each cubic metre of the forest occupies a larger land area (and land value) in the beginning of the time interval than later (because of growth), it is more important to harvest rapidly in the beginning of the period than later. The *qualitative* result is found in Appendix M. 2, Remark 2. This

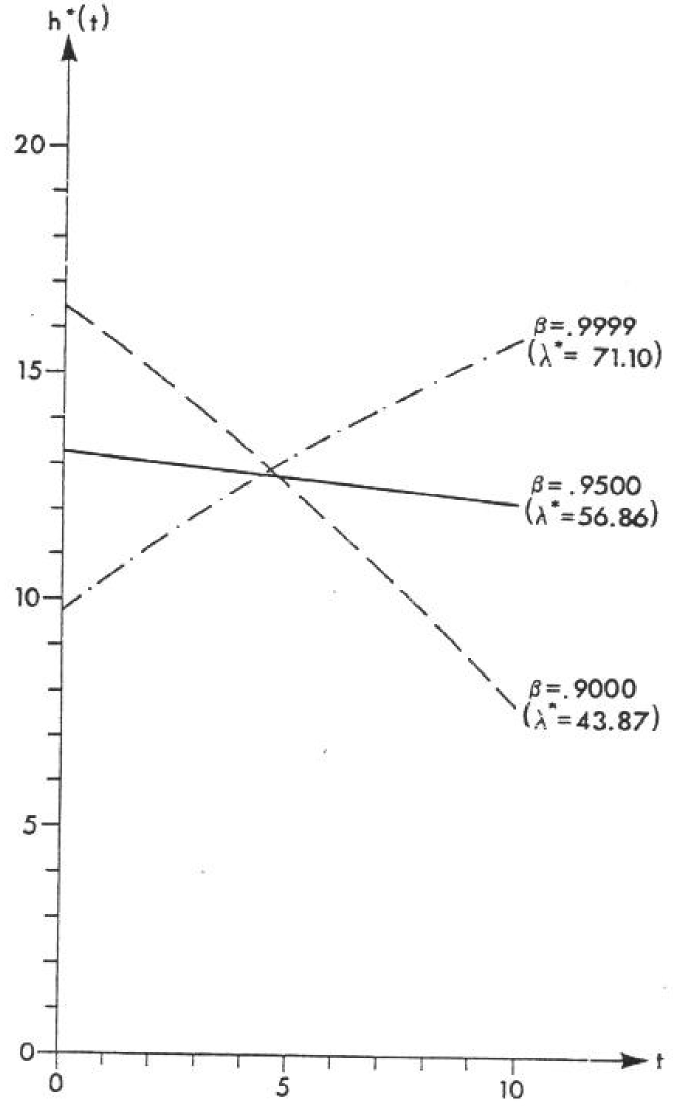

Fig. 7. The optimal harvest $h^*(t)$ as a function of the rate of interest ($\beta = e^{-r}$) CASE 2, a renewable resource. $\pi(h) = -F + 100h - 2h^2$, $\alpha = .95$, $Q_0 = 100$, $G = 10$, $T = 10$

is found in the graph: The optimal harvest level for $\beta = .95$ is slowly declining. The marginal resource value λ^* is a decreasing function of the rate of interest (an increasing function of β) since discounting reduces the value of future harvesting.

In Fig. 8, the influence of the growth rate is highlighted. In short: If the growth rate increases (α decreases), then the total harvest over the period, $\int_0^T h(t)\,dt$, increases. If the marginal value of harvesting in different periods would not be negatively affected by the increasing harvest level, then increasing relative growth would imply that a larger proportion of the resource should be saved for the future. As shown in Fig. 8, this is also the general tendency. However, since π is strictly concave, this tendency does not hold completely. It is not optimal to increase the harvest level in a particular period too much. ($h^*(0)$ decreases as α decreases from .9999 to .95. Then, however, $h^*(0)$ increases again when α takes the value .90.) The marginal value of the resource, λ^* is an increasing function of the growth rate for low values of the growth rate. However, if the growth rate increases very much, (α decreases very much), then, because of concavity in π, the marginal resource value λ^* decreases again.

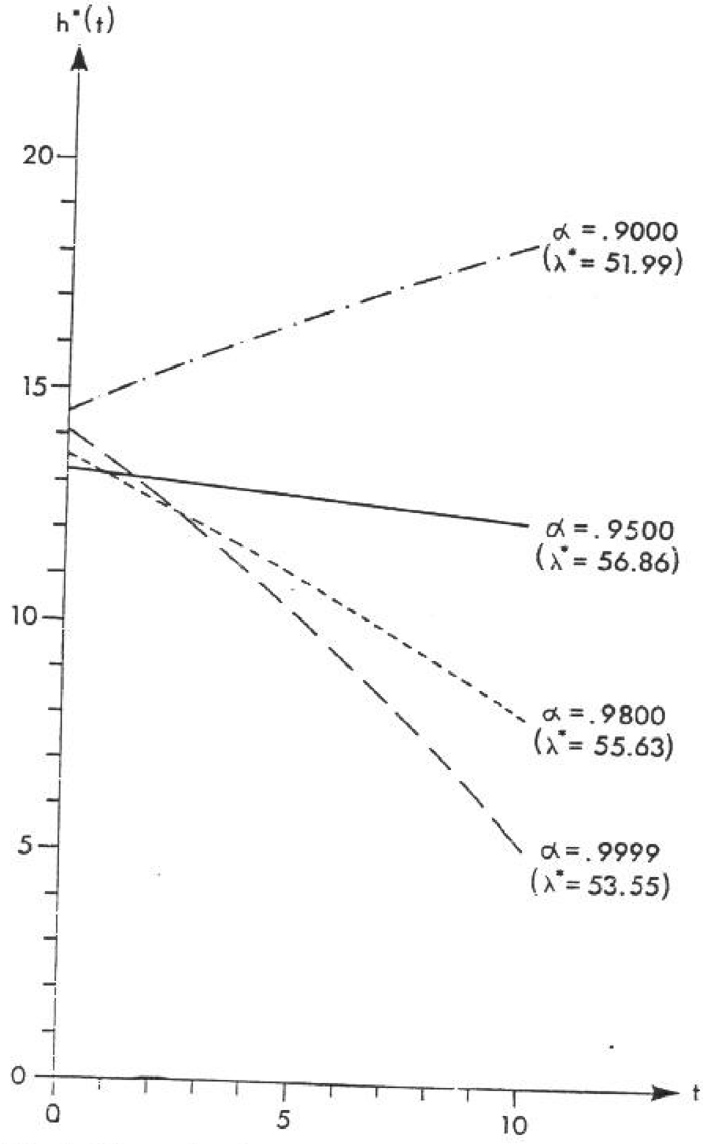

Fig. 8. The optimal harvest $h^*(t)$ as a function of the growth rate. CASE 2, a renewable resource. $\pi(h) = -F + 100h - 2h^2$, $\beta = .95$, $Q_0 = 100$, $G = 10$, $T = 10$

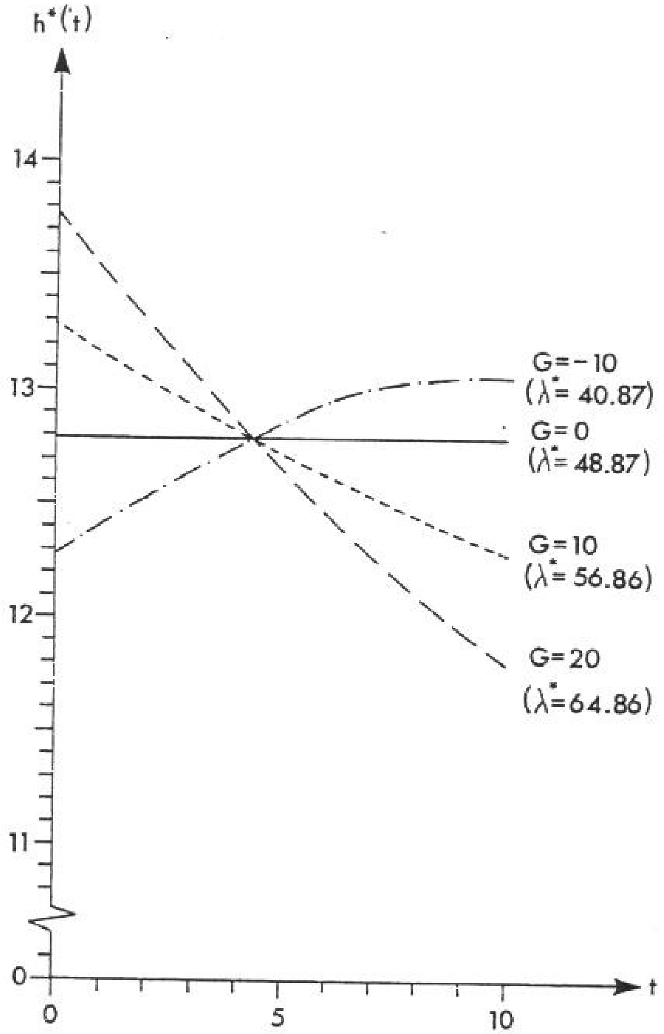

Fig. 9. The optimal harvest $h^*(t)$ as a function of the land value. CASE 2, a renewable resource. $\pi(h) = -F + 100h - 2h^2$, $\alpha = .95$, $\beta = .95$, $Q_0 = 100$, $T = 10$

The influence of the land value, finally, is demonstrated in Fig. 9. Compare appendix M. 2, Remark 2, where the *qualitative* result is shown. In all of the graphs, the rate of interest corresponds to the growth rate of the forest. Hence, if there is no land value, the optimal harvest path is a constant. As discussed also in connection to Fig. 7, a strictly positive land value will increase the optimal harvest level in the beginning of the period and decrease it in the end. The magnitude of this effect is shown in the graph. A special case is also illustrated: If the land owner is obliged to undertake expensive reforestration activities (that make the present value of future forest generations negative), this is consistent with a strictly negative land value. Then, of course, it is optimal to reduce the harvest level in the beginning of the period and to increase it in the future.

6. Discussion

This paper has given the following results;

a. A mathematically convenient approach to continuous (and discrete) time optimization of intertemporal extraction (harvesting) has been developed.

b. A computer program based on the analytical derivations has been constructed.
c. The nonrenewable resource extraction problem has been solved and discussed through the new method.
d. A new version of the forest harvesting problem has been defined, solved and discussed.

Some possible and important future extensions of the theory are;

e. to make the planning horizon endogenous,
f. to generalize the model assumptions concerning prices, costs and growth,
g. to derive optimal solutions to modified versions of the problems explicitly,
h. and finally to estimate the robustness of the derived results. (Are the solutions very sensitive to the assumptions concerning deterministic relationships? If we introduce stochastic variables in the system which are sequentially observed, adaptive optimization is the relevant approach.)

Hopefully the reader finds new areas of application of the new method.

M. Mathematical Appendix

M.1. The Problem in Discrete Time

$$\max \Pi = \sum_{t=0}^{T} e^{-rt} ((\text{HARVEST NET REVENUE}_t) + (\text{RELEASED LAND VALUE}_t))$$

$$\text{s.t.} \sum_{t=0}^{T} (\text{HARVEST AREA}_t) \overset{(\leq)}{=} (\text{TOTAL AREA})$$

It will everywhere be implicitly assumed that the optimal solution is an "*interior solution*", which means that the harvest level is strictly positive in every time period. If we make use of the parameter definitions (from Section 3), we get:

$$\max \Pi = \sum_{t=0}^{T} e^{-rt} ((ah_t + bh_t^2) + \Omega_t), \ (\Omega_t = g\alpha^t h_t)$$

$$\text{s.t.} \sum_{t=0}^{T} \alpha^t h_t \overset{(\leq)}{=} Q_0$$

The Lagrange function becomes:

$$L = \sum_{t=0}^{T} \beta^t((a + g\alpha^t) h_t + bh_t^2) + \lambda \left(Q_0 - \sum_{t=0}^{T} \alpha^t h_t\right)$$

The first order optimum conditions are;

$$\frac{\partial L}{\partial \lambda} = Q_0 - \sum_{t=0}^{T} \alpha^t h_t = 0$$

$$\frac{\partial L}{\partial h_t} = \beta^t(a + g\alpha^t + 2bh_t) - \lambda\alpha^t = 0 \quad (\text{for } 0 \leq t \leq T)$$

M.2. The Optimal Solution in Discrete Time

Clearly, $\frac{\partial L}{\partial h_t} = 0$ implies that:

$$h_t^* = \frac{\lambda^*(\alpha/\beta)^t - a - g\alpha^t}{2b} \quad \text{(where * denotes the optimal value).}$$

Hence, we must determine λ^*. Then, however, h_t^* is determined for all values of t! Now, it is easy to determine λ^* via the resource constraint $\left(\frac{\partial L}{\partial \lambda} = 0\right)$.

$$Q_0 = \sum \alpha^t h_t^*$$

$$Q_0 = \sum \alpha^t \left[\frac{\lambda^*(\alpha/\beta)^t - a - g\alpha^t}{2b}\right]$$

$$2b\, Q_0 = \lambda^* \sum (\alpha^2/\beta)^t - a \sum \alpha^t - g \sum \alpha^{2t}$$

λ^* can now be determined from:

$$\lambda^* = \frac{2bQ_0 + a \sum \alpha^t + g \sum \alpha^{2t}}{\sum (\alpha^2/\beta)^t}.$$

Hence, after summing the series, the solution is explicitly obtained:

$$\lambda^* = \frac{2bQ_0 + a\frac{1-\alpha^{T+1}}{1-\alpha} + g\frac{1-(\alpha^2)^{T+1}}{1-\alpha^2}}{\frac{1-(\alpha^2/\beta)^{T+1}}{1-(\alpha^2/\beta)}}.$$

Definition.

$$\Delta h_t^* = (h_{t+1}^* - h_t^*) = (2b)^{-1}\left[\lambda^*((\alpha/\beta)^{t+1} - (\alpha/\beta)^t) - g(\alpha^{t+1} - \alpha^t)\right]$$

Remark 1. If $b < 0$, $\lambda^* > 0$ and $g = 0$, then it follows that;

$$\begin{bmatrix} a > \beta \\ \alpha = \beta \\ \alpha < \beta \end{bmatrix} \Rightarrow \Delta h_t^* \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0$$

Remark 2. If $b < 0$ and $\alpha = \beta < 0$, then it follows that;

$$g \begin{bmatrix} > \\ = \\ < \end{bmatrix} 0 \Rightarrow \Delta h_t^* \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0$$

M.3. Second Order Maximum Conditions

The problem is to maximize a function, $\Pi(h)$, where h denotes the vector containing h_t for all t such that $0 \le t \le T$ subject to a resource constraint. The most general approach to the study of the second order maximum condition is to investigate the signs of the bordered Hessian matrix (compare CHIANG [3, Table 12.1]). However, since in this application, we know that the restriction is linear in the relevant space, it is well known (CHIANG [3, p. 718]) that the resulting feasible area is a convex set and the constraint qualification of nonlinear programming is met. Then, if the objective function Π is (strictly) concave, the solution obtained from the first order optimum conditions is a (unique) global maximum (Compare CHIANG [3, p. 722].). Define $[D_T]$ as;

$$[D_T] = \begin{bmatrix} \Pi_{h_0h_0} & \Pi_{h_0h_1} & \vdots & \Pi_{h_0h_T} \\ \Pi_{h_1h_0} & \Pi_{h_1h_1} & \vdots & \Pi_{h_1h_T} \\ \cdots & \cdots & & \cdots \\ \Pi_{h_Th_0} & \Pi_{h_Th_1} & \vdots & \Pi_{h_Th_T} \end{bmatrix}.$$

Then, $\Pi(h)$ is strictly concave if;

$$|\Pi_{h_0h_0}| < 0, \begin{vmatrix} \Pi_{h_0h_0} & \Pi_{h_0h_1} \\ \Pi_{h_1h_0} & \Pi_{h_1h_1} \end{vmatrix} > 0, \ldots$$

$$\ldots, (-1)^t \begin{vmatrix} \Pi_{h_0h_0} & \Pi_{h_0h_1} & \vdots & \Pi_{h_0h_t} \\ \Pi_{h_1h_0} & \Pi_{h_1h_1} & \vdots & \Pi_{h_1h_t} \\ \cdots & \cdots & & \cdots \\ \Pi_{h_th_0} & \Pi_{h_th_1} & \vdots & \Pi_{h_th_t} \end{vmatrix} > 0, \ldots, (-1)^T |D_T| > 0.$$

However, in the applications of this paper and in most intertemporal economic problems, the test of concavity is very easy. The objective function Π is a weighted sum of the profits in different periods, π. Hence, if the profit, π, is a concave function of the harvest level in each period, then Π is concave. In the applications in this paper, it is easy to find that Π is strictly concave. Thus, we always obtain a unique global maximum through the first order optimum conditions.

M.4. From Discrete to Continuous Time

Now, the discrete time problem will be slightly modified. We replace the original assumption of T periods by vT periods where v is a positive integer greater than 1. The new periods are denoted by s. We still assume that the total time interval under consideration is $(0, T)$, measured in the original time scale. Each period in the new time scale (s) represents $1/v$ periods in the original scale (t). Since each period is shorter than before, the harvest in each period contributes less to the objective function and influences the resource constraint less.

The modified problem is (the land value is not included in the following derivations):

$$\max \Pi = \sum_{s=0}^{vT} e^{-r(s/v)} \pi(h_{(s/v)}) \left(\frac{1}{v}\right)$$

$$\text{s.t.} \sum_{s=0}^{vT} \alpha^{(s/v)} h_{(s/v)} \left(\frac{1}{v}\right) \overset{(\leq)}{\leqq} Q_0$$

Clearly, we may replace $1/v$ by Δt. As $\Delta t \to 0$, the objective function and the restriction may be described as:

$$\max \Pi = \int_0^T e^{-rt} \pi(h(t)) \, dt \quad (h(t) \text{ corresponds to } h_t)$$

$$\text{s.t.} \int_0^T \alpha^t h(t) \, dt \overset{(\leq)}{\leqq} Q_0$$

M.5. The Optimal Solution in Continuous Time

The problem in continuous time, including the land value, becomes:

$$\text{Max } \Pi = \int_0^T e^{-rt}((a + g\alpha^t)\, h(t) + bh^2(t))\, dt$$

$$\text{s.t. } \int_0^T \alpha^t h(t)\, dt \overset{(<)}{=} Q_0$$

We use the method described for the discrete time case on the modified (shorter time periods) discrete time problem, to determine $h^*(t)$. Clearly, the formula becomes the same as in the original discrete time case also when the periods represent time intervals that approach zero. However, λ^* will take a slightly different value, which in turn makes $h^*(t)$ slightly different from h_t^*:

$$h^*(t) = \frac{\lambda^*(\alpha/\beta)^t - a - g\alpha^t}{2b}.$$

A convenient form of the expression in the following calculations is:

$$h^*(t) = (2b)^{-1}\,[\lambda^*\, e^{t*LN(\alpha/\beta)} - a - g\, e^{t*LN(\alpha)}].$$

Now, we may determine the value of λ^* through the resource constraint and integration over time:

$$Q_0 = \int \alpha^t h(t)\, dt$$

$$2bQ_0 = -a \int (\alpha^t)\, dt + \lambda^* \int (\alpha^t e^{t*LN(\alpha/\beta)})\, dt - g \int (\alpha^t e^{t*LN(\alpha)})\, dt$$

$$2bQ_0 = -a \int (e^{t*LN(\alpha)})\, dt + \lambda^* \int (e^{(LN(\alpha/\beta)+LN(\alpha))t})\, dt - g \int (e^{2t*LN(\alpha)})\, dt$$

From this, we may solve λ^*:

$$\lambda^* = \frac{2bQ_0 + a \int (e^{t*LN(\alpha)})\, dt + g \int (e^{2t*LN(\alpha)})\, dt}{\int (e^{(LN(\alpha/\beta)+LN(\alpha))t})\, dt}.$$

Making use of the limits of integration 0 and T, we find that:

$$\lambda^* = \frac{2bQ_0 + a\left(\left.\frac{e^{t*LN(\alpha)}}{LN(\alpha)}\right]_0^T\right) + g\left(\left.\frac{e^{2t*LN(\alpha)}}{2LN(\alpha)}\right]_0^T\right)}{\left(\left.\frac{e^{LN(\alpha^2/\beta)t}}{LN(\alpha^2/\beta)}\right]_0^T\right)}$$

$$\lambda^* = \frac{2bQ_0 + \frac{a}{LN(\alpha)}(e^{LN(\alpha)T} - 1) + \frac{g}{2LN(\alpha)}(e^{2LN(\alpha)T} - 1)}{\frac{e^{LN(\alpha^2/\beta)T} - 1}{LN(\alpha^2/\beta)}}$$

$$\lambda^* = \frac{2bQ_0 + \frac{a}{LN(\alpha)}(\alpha^T - 1) + \frac{g}{2LN(\alpha)}(\alpha^{2T} - 1)}{\frac{(\alpha^2/\beta)^T - 1}{LN(\alpha^2/\beta)}}.$$

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N. Numerical Appendix

This program is constructed from the equations derived in the mathematical appendix.

```
3 LPRINT CHR$(27); "G"
5 LPRINT" "
6 LPRINT"*****************************************"
10 LPRINT"          OPTIMAL EXTRACTION IN CONTINUOUS TIME"
20 LPRINT"          LOHMANDER PETER 88-02-18"
25 LPRINT" "
30 INPUT" A = ", A
40 INPUT" B = ", B
45 INPUT" G = ", G
50 INPUT" ALFA = ", ALFA
60 INPUT" BETA = ", BETA
70 INPUT" Q0 = ", Q0
80 INPUT" T = ", T
85 INPUT" PRICE INTERCEPT = ", K1
86 INPUT" PRICE SLOPE = ", K2
90 LPRINT"          A = "; A;" B = "; B;" ALFA = "; ALFA;" BETA = "; BETA
100 LPRINT"          Q0 = "; Q0;" G = "; G;" T = "; T
102 LPRINT"          PRICE INTERCEPT = "; K1;" PRICE SLOPE = "; K2
105 LPRINT" "
110 L = 2*B*Q0 + A/(LOG(ALFA))*(ALFA^T - 1)
115 L = L + G/(2*LOG(ALFA))*(ALFA^(2*T) - 1)
120 L = L*LOG(ALFA^2/BETA)/((ALFA^2/BETA)^T - 1)
125 LPRINT"          LAMBDA, THE MARGINAL RESOURCE VALUE = "; L
126 LPRINT" "
200 FOR S = 0 TO T
210 H = (L*(ALFA/BETA)^S - A - G*ALFA^S)/(2*B)
220 P = K1 + K2*H
225 LPRINT"          YEAR = "; S;" OPTIMAL EXTRACTION = "; H;" PRICE =
          "; P
230 NEXT S
240 STOP
```

```
*******************************************
        OPTIMAL EXTRACTION IN CONTINUOUS TIME
        LOHMANDER PETER 88-02-18

        A = 100  B = -2  ALFA = .9999  BETA = .95
        Q0 = 100  G = 0  T = 10
        PRICE INTERCEPT = 150  PRICE SLOPE = -1

        LAMBDA, THE MARGINAL RESOURCE VALUE = 45.90353

        YEAR = 0    OPTIMAL EXTRACTION = 13.52412   PRICE = 136.4759
        YEAR = 1    OPTIMAL EXTRACTION = 12.92133   PRICE = 137.0787
        YEAR = 2    OPTIMAL EXTRACTION = 12.28688   PRICE = 137.7131
        YEAR = 3    OPTIMAL EXTRACTION = 11.61911   PRICE = 138.3809
        YEAR = 4    OPTIMAL EXTRACTION = 10.91626   PRICE = 139.0837
        YEAR = 5    OPTIMAL EXTRACTION = 10.17649   PRICE = 139.8235
```

YEAR = 6	OPTIMAL EXTRACTION = 9.397868	PRICE = 140.6021
YEAR = 7	OPTIMAL EXTRACTION = 8.578346	PRICE = 141.4217
YEAR = 8	OPTIMAL EXTRACTION = 7.715777	PRICE = 142.2842
YEAR = 9	OPTIMAL EXTRACTION = 6.8079	PRICE = 143.1921
YEAR = 10	OPTIMAL EXTRACTION = 5.852335	PRICE = 144.1477

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