Optimal Deployment

by
Peter Lohmander

ICSTC-2024

Tenth International Conference on Statistics for the Twenty-First Century

December 13 - 16, 2024, Special Invited Talk.

Organized by International Statistics Fraternity (ISF) in collaboration with the Department of Statistics, University of Kerala and School of Physical and Mathematical Sciences at the University of Kerala and the American Statistical Association.

Version 241211_1448

Peter Lohmander
Prof Dr

Peter Lohmander Optimal Solutions, Sweden

Lohmander, P., Optimal Deployment, December 13 - 16, 2024, Special Invited Talk, ICSTC-2024, Tenth International Conference on Statistics for the Twenty-First Century.

Abstract

This study focuses on the optimal deployment problem, and determines the optimal size of a military force to send to the battle field. The decision is optimized, based on an objective function, that considers the cost of deployment, the cost of the time it takes to win the battle, and the costs of killed and wounded soldiers with equipment. The cost of deployment is modeled as an explicit function of the number of deployed troops and the value of a victory with access to a free territory, is modeled as a function of the length of the time it takes to win the battle. The cost of lost troops and equipment, is a function of the size of the reduction of these lives and resources. An objective function, based on these values and costs, is optimized, under different parameter assumptions. The battle dynamics is modeled via the Lanchester differential equation system based on the principles of directed fire. First, the deterministic problem is solved analytically, via derivations and comparative statics analysis. General mathematical results are reported, including the directions of changes of the optimal deployment decisions, under the influence of alternative types of parameter changes. Then, the first order optimum condition from the analytical model, in combination with numerically specified parameter values, is used to determine optimal values of the levels of deployment in different situations. A concrete numerical case, based on documented facts from the Battle of Iwo Jima, during WW II, is analyzed, and the optimal US deployment decisions are determined under different assumptions. The known attrition coefficients of both armies, from USA and Japan, and the initial size of the Japanese force, are parameters. The analysis is also based on some parameters without empirical documentation, that are necessary to include to make optimization possible. These parameter values are motivated in the text. The optimal solutions are found via Newton-Raphson iteration. Finally, a stochastic version of the optimal deployment problem is defined. The attrition parameters are considered as stochastic, before the deployment decisions have been made. The attrition parameters of the two armies have the same expected values as in the deterministic analysis, are independent of each other, have correlation zero, and have relative standard deviations of 20%. All possible deployment decisions, with 5000 units intervals, from 0 to 150000 troops, are investigated, and the optimal decisions are selected. The analytical, and the two numerical, methods, all show that the optimal deployment level is a decreasing function of the marginal deployment cost, an increasing function of the marginal cost of the time to win the battle, an increasing function of the marginal cost of killed and wounded soldiers and lost equipment, an increasing function of the initial size of the opposing army, an increasing function of the efficiency of the soldiers in the opposing army and a decreasing function of the efficiency of the soldiers in the deployed army. With stochastic attrition parameters, the stochastic model also shows that the probability to win the battle is an increasing function of the size of the deployed army. When the optimal deployment level is selected, the probability of a victory is usually less than 100%, since it would be too expensive to guarantee a victory with 100% probability.

This study focuses on the optimal deployment problem, and determines the optimal size of a military force to send to the battle field.

The decision is optimized, based on an objective function, that considers the cost of deployment,

the cost of the time it takes to win the battle, and

the costs of killed and wounded soldiers with equipment.

The cost of deployment is modeled as an explicit function of the number of deployed troops and the value of a victory with access to a free territory, is modeled as a function of the length of the time it takes to win the battle. The cost of lost troops and equipment, is a function of the size of the reduction of these lives and resources.

An objective function, based on these values and costs, is optimized, under different parameter assumptions.

The battle dynamics is modeled via the Lanchester differential equation system based on the principles of directed fire.

First, the deterministic problem is solved analytically, via derivations and comparative statics analysis. General mathematical results are reported, including the directions of changes of the optimal deployment decisions, under the influence of alternative types of parameter changes.

Then, the first order optimum condition from the analytical model, in combination with numerically specified parameter values, is used to determine optimal values of the levels of deployment in different situations.

A concrete numerical case, based on documented facts from the Battle of Iwo Jima, during WW II, is analyzed, and the optimal US deployment decisions are determined under different assumptions.

The known attrition coefficients of both armies, from USA and Japan, and the initial size of the Japanese force, are parameters. The analysis is also based on some parameters without empirical documentation, that are necessary to include to make optimization possible. These parameter values are motivated in the text.

The optimal solutions are found via Newton- Raphson iteration.

Finally, a stochastic version of the optimal deployment problem is defined. The attrition parameters are considered as stochastic, before the deployment decisions have been made.

The attrition parameters of the two armies have the same expected values as in the deterministic analysis, are independent of each other, have correlation zero, and have relative standard deviations of 20%.

All possible deployment decisions, with 5000 units intervals, from 0 to 150000 troops, are investigated, and the optimal decisions are selected.

Background information:

Lohmander, P., Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions,
WSTA-2024, Recent Trends in Statistical Theory and Applications-2024 (WSTA-2024)
June 29 – July 02, 2024, Invited Talk.
Organized by the Department of Statistics, School of Physical and Mathematical Sciences, University of Kerala, Trivandrum in association with Indian Society for Probability and Statistics (ISPS) and Kerala Statistical Association (KSA).
http://www.Lohmander.com/PL_WSTA_2024.pdf
http://www.Lohmander.com/PL_WSTA_2024.pptx

Lohmander, P., Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions,
Journal of Statistics and Computer Science, Vol. 3, Issue 1, 2024, pp. 51-78.
https://www.arfjournals.com/jscs/issue/322
https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

Lohmander, P. Optimal Deployment. Preprints 2024, 2024021265.
https://doi.org/10.20944/preprints202402.1265.v1
https://www.preprints.org/manuscript/202402.1265/v1/download

The Lanchester differential equations:

$$\begin{cases} \dot{x} = -ay & (1.a) \\ \dot{y} = -bx & (1.b) \end{cases}$$

$$a > 0, b > 0, x > 0, y > 0$$

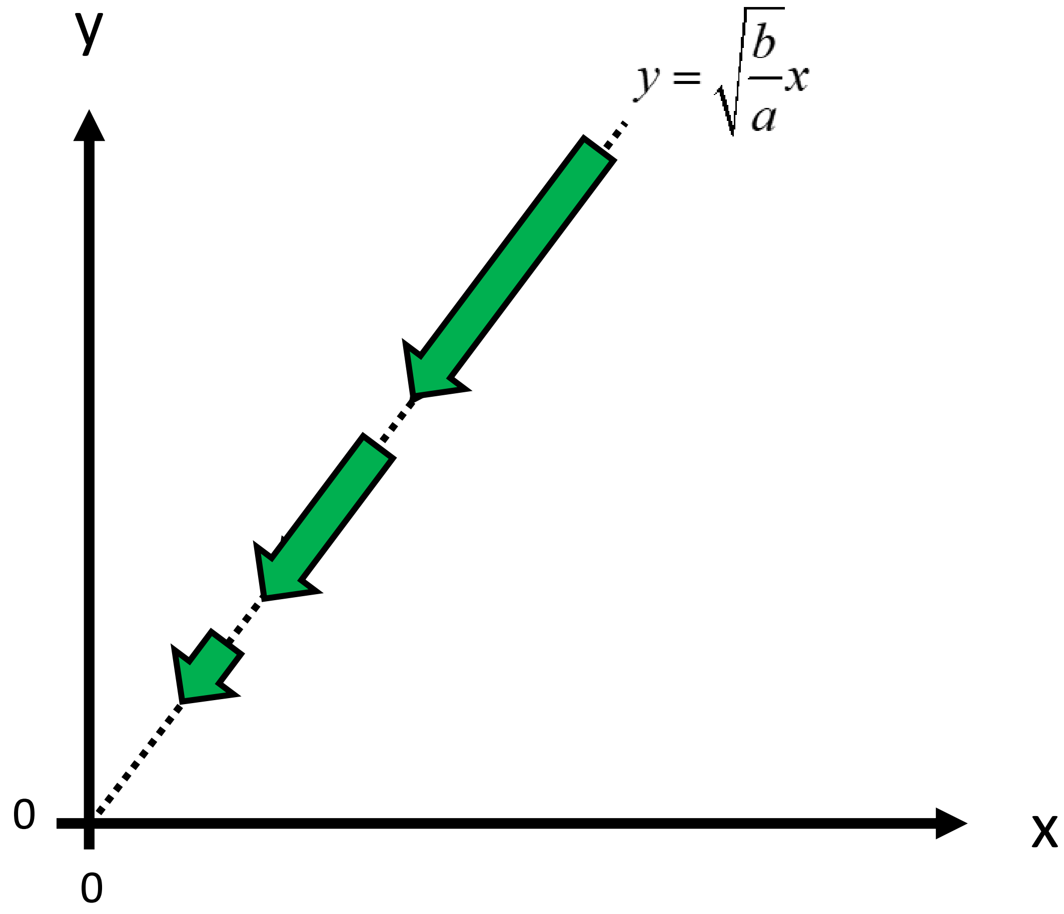


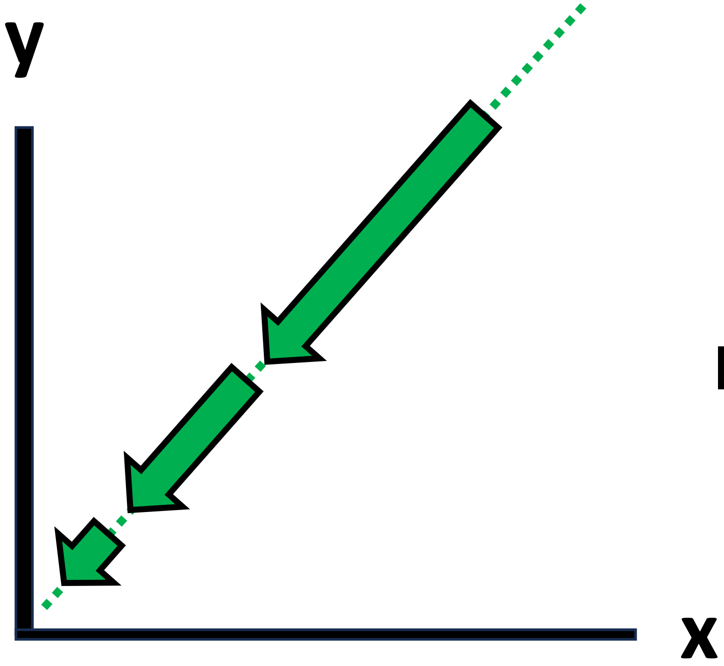
Figure 1.

The time path of (x, y) in the special case, when $bx^2=ay^2$.

Why is that the case?

$$\begin{cases} \dfrac{\dot{x}}{x} = \dfrac{-ay}{x} & (2.a) \\ \dfrac{\dot{y}}{y} = \dfrac{-bx}{y} & (2.b) \end{cases}$$

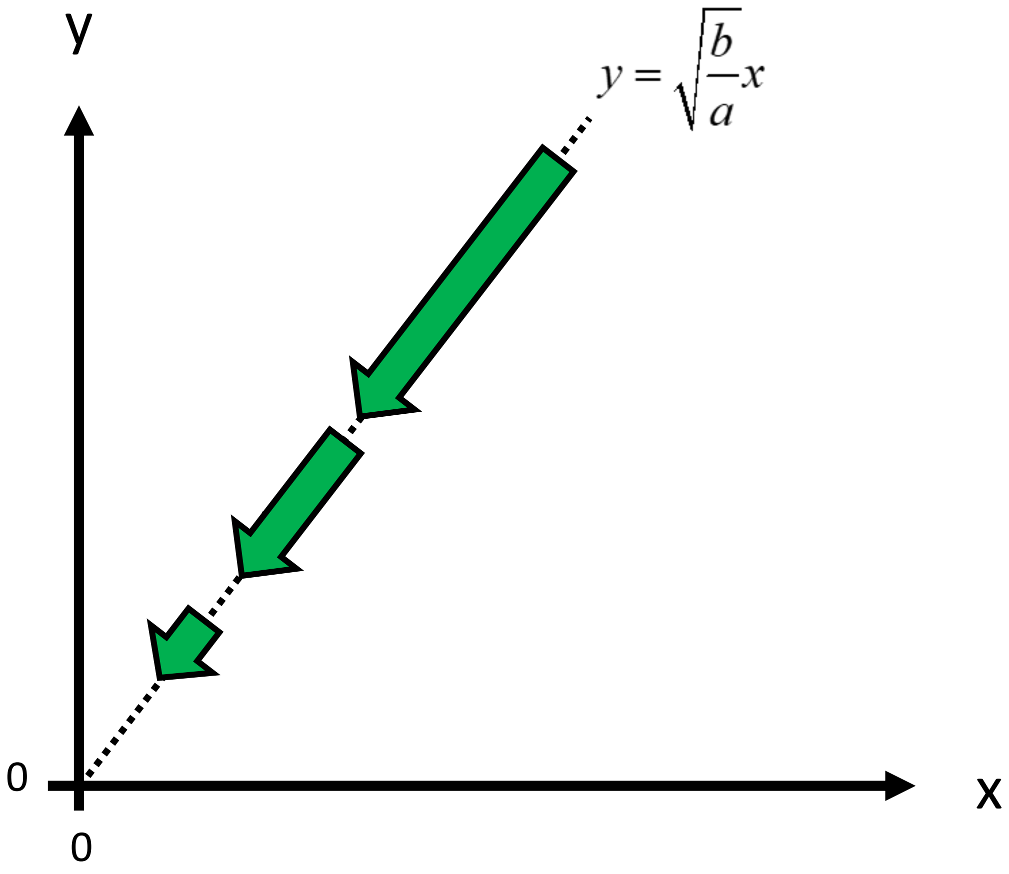
$$a > 0, b > 0, x > 0, y > 0$$



$$\left(\frac{\dot{x}}{x} = \frac{\dot{y}}{y}\right) \Rightarrow \left(\frac{-ay}{x} = \frac{-bx}{y}\right)$$

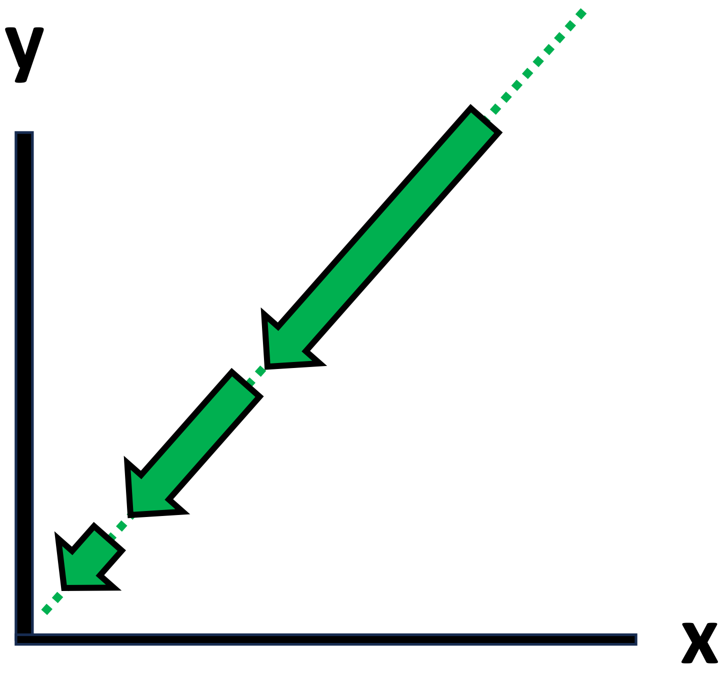
$$\frac{bx}{y} = \frac{ay}{x}$$

$$x^2 = \frac{a}{b} y^2$$



$$y = \sqrt{\frac{b}{a}}x$$

$$x = \sqrt{\frac{a}{b}}y$$



$$\begin{cases} \dot{x} = -ay & (1.a) \\ \dot{y} = -bx & (1.b) \end{cases}$$

$$a > 0, b > 0, x > 0, y > 0$$

$$x^2 = \frac{a}{b} y^2$$

$$x = \sqrt{\frac{a}{b}} y \Rightarrow \lim_{t \to \infty} \dot{x} = 0$$

$$y = \sqrt{\frac{b}{a}} x \Rightarrow \lim_{t \to \infty} \dot{y} = 0$$

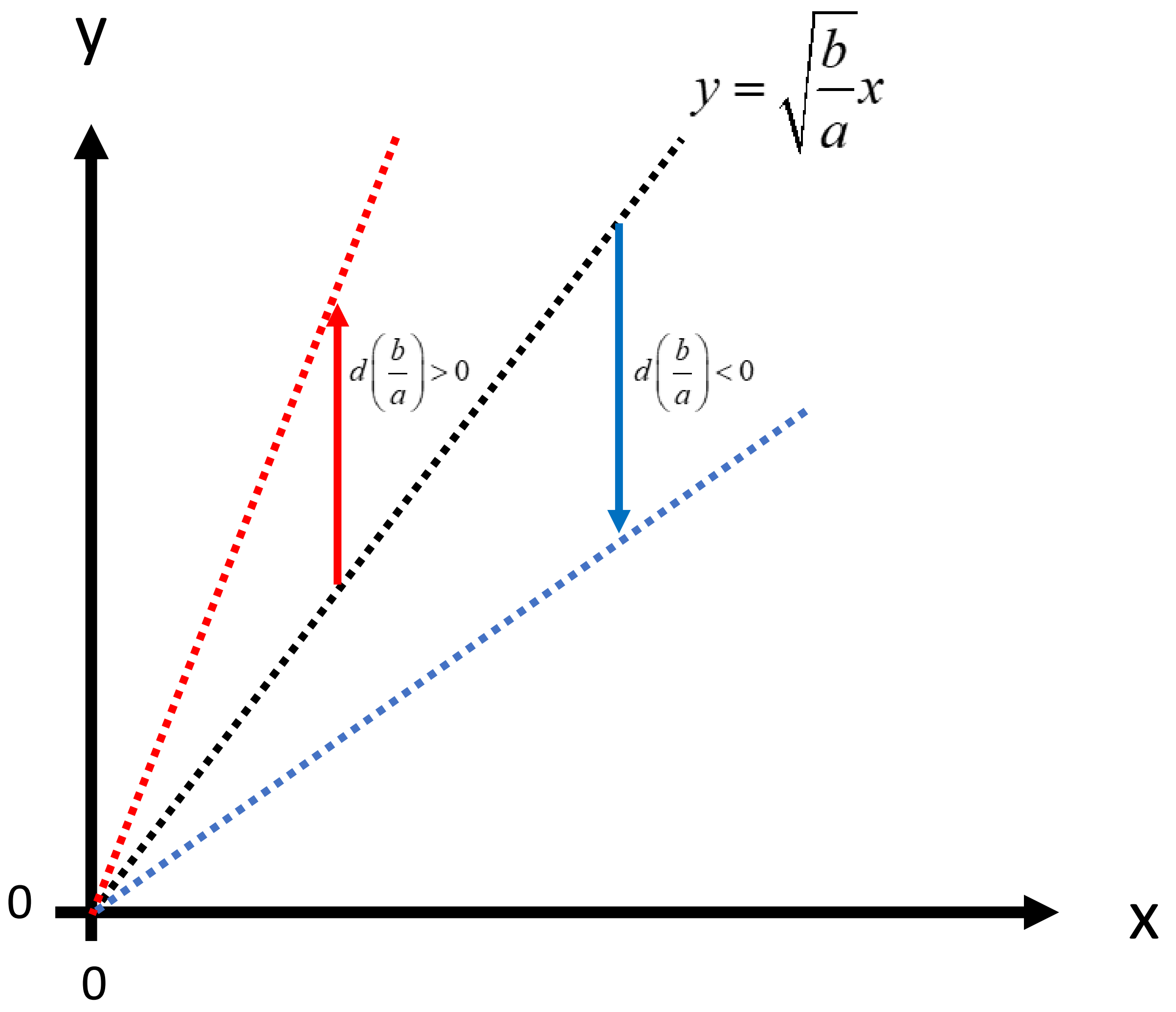


Figure 2.

The time path of (x,y) in the special case, when $bx^2=ay^2$, is a function of the ratio b/a. The graph shows how the time path changes if the ratio b/a increases or decreases.

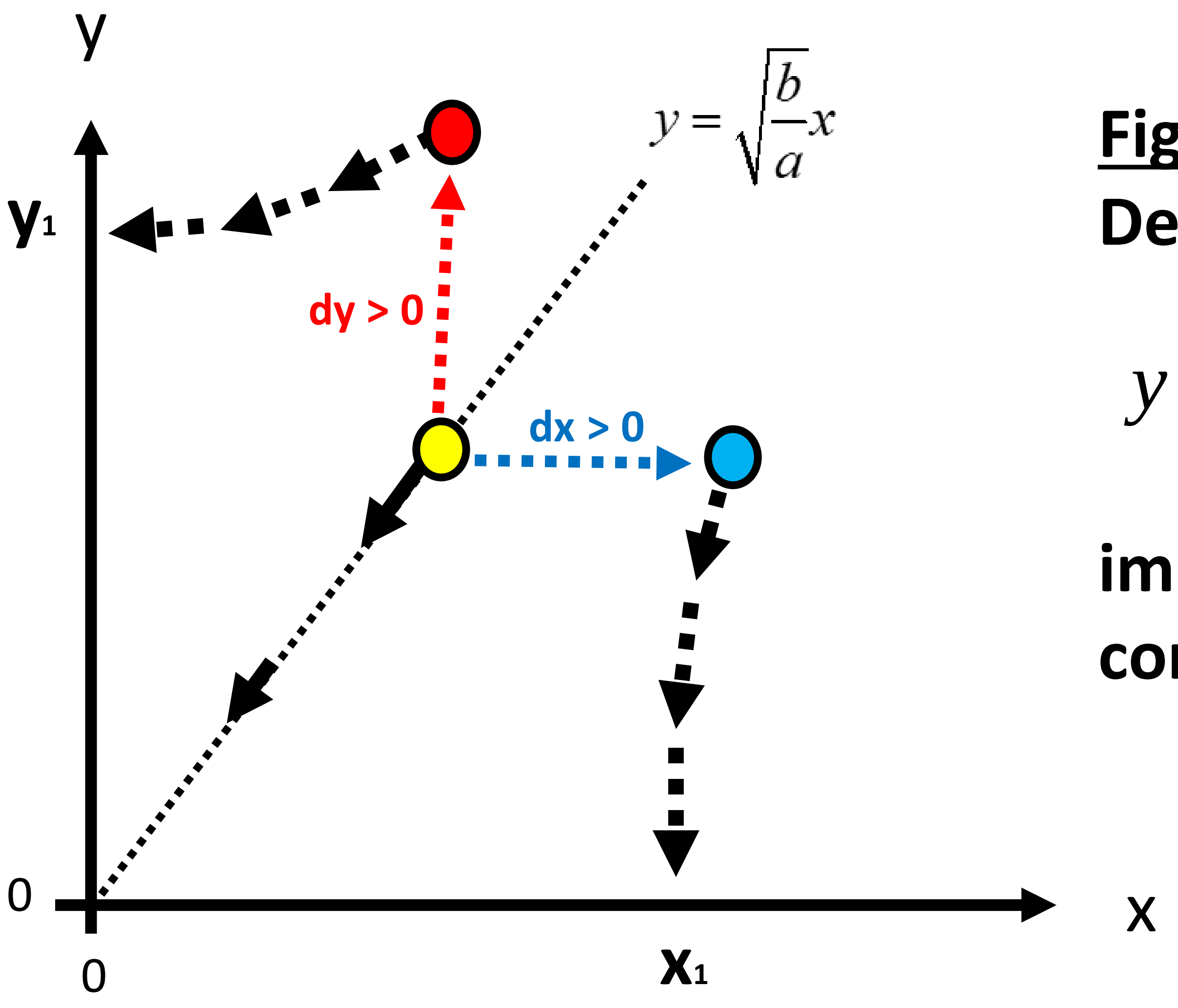


Figure 3.

Deviations from the line

$$y=\sqrt{\frac{b}{a}}x$$

imply that (x, y) will not converge to origo.

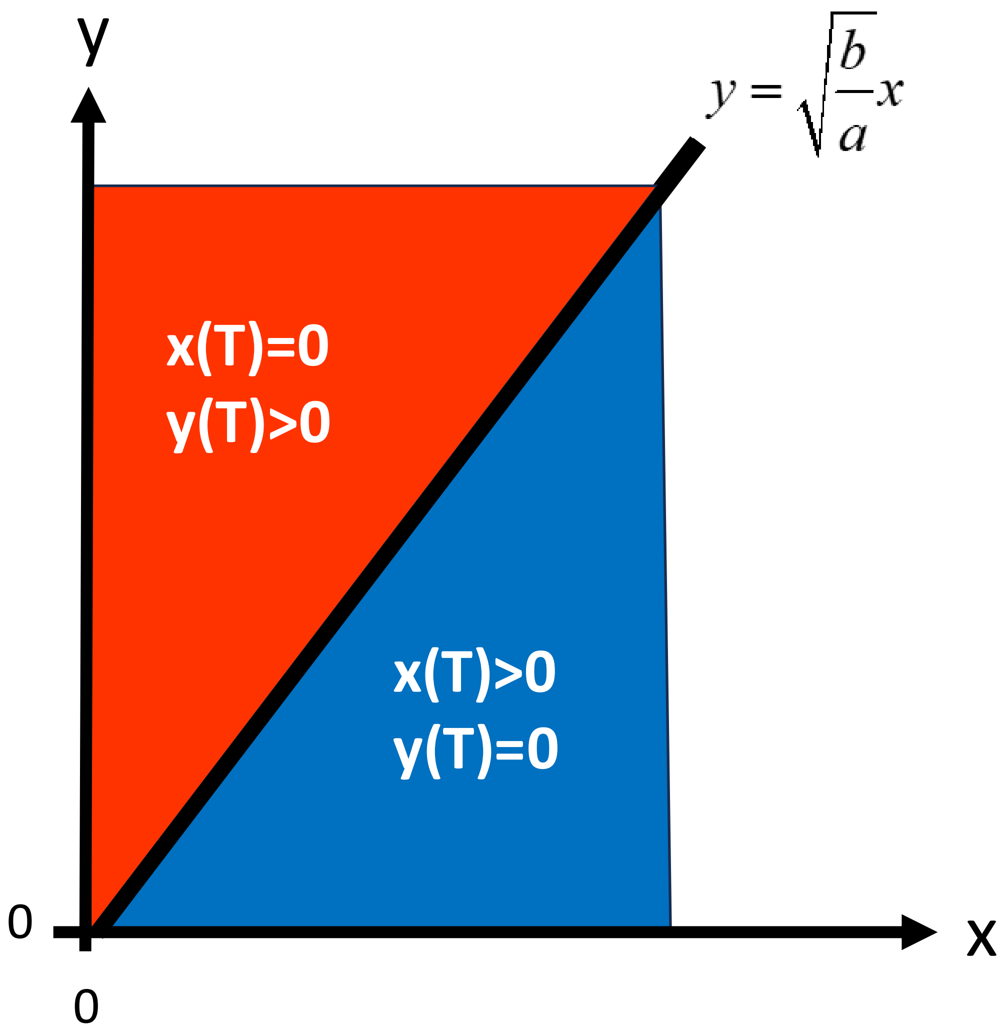
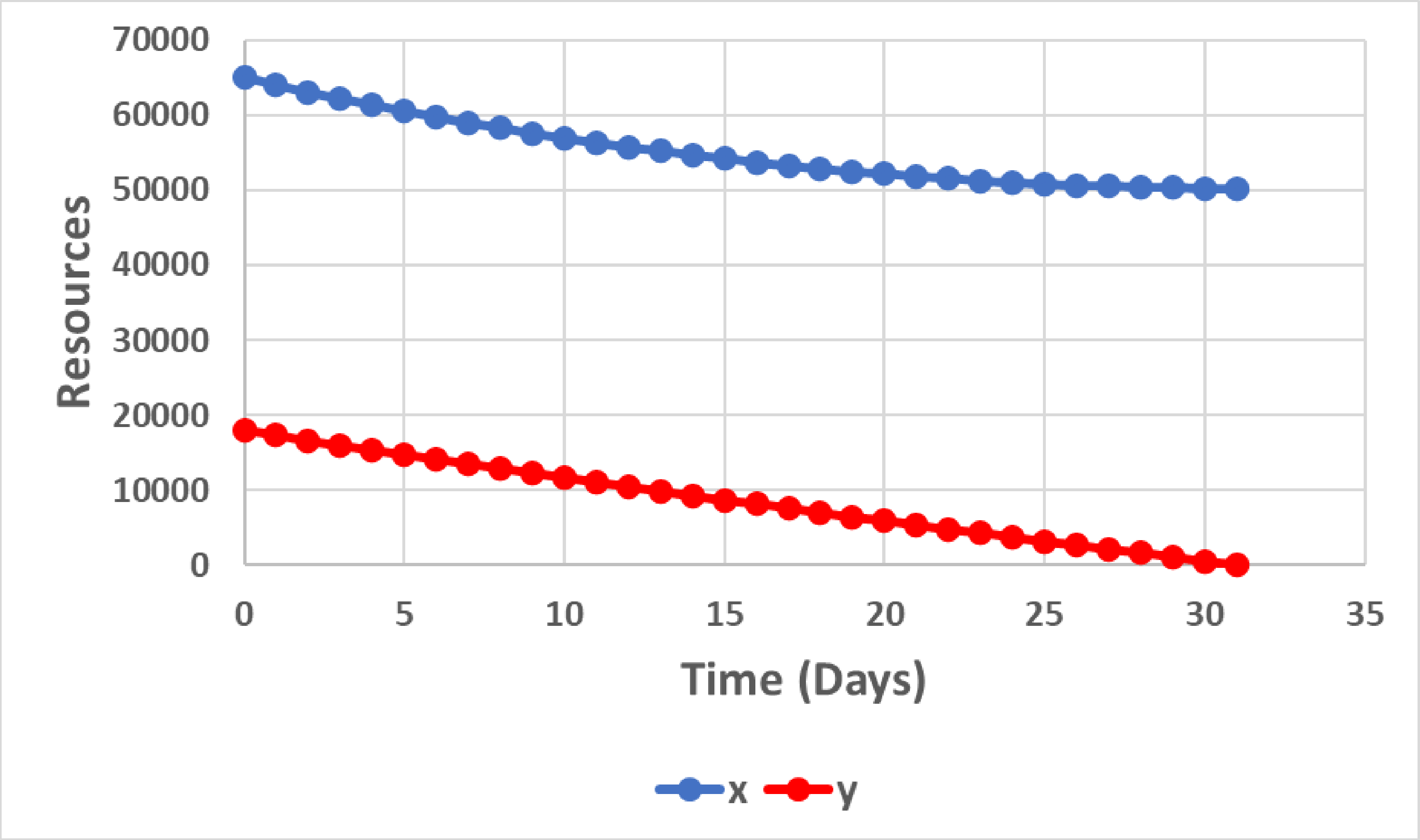


Figure 4.

T is the point in time when x or y equals zero. If (x, y) at some point in time, t, such that t<T, is found in the blue sector, then x(T)>0 and y(T)=0. If (x, y) at some point in time, t, such that t<T, is found in the red sector, then x(T)=0 and y(T)>0.



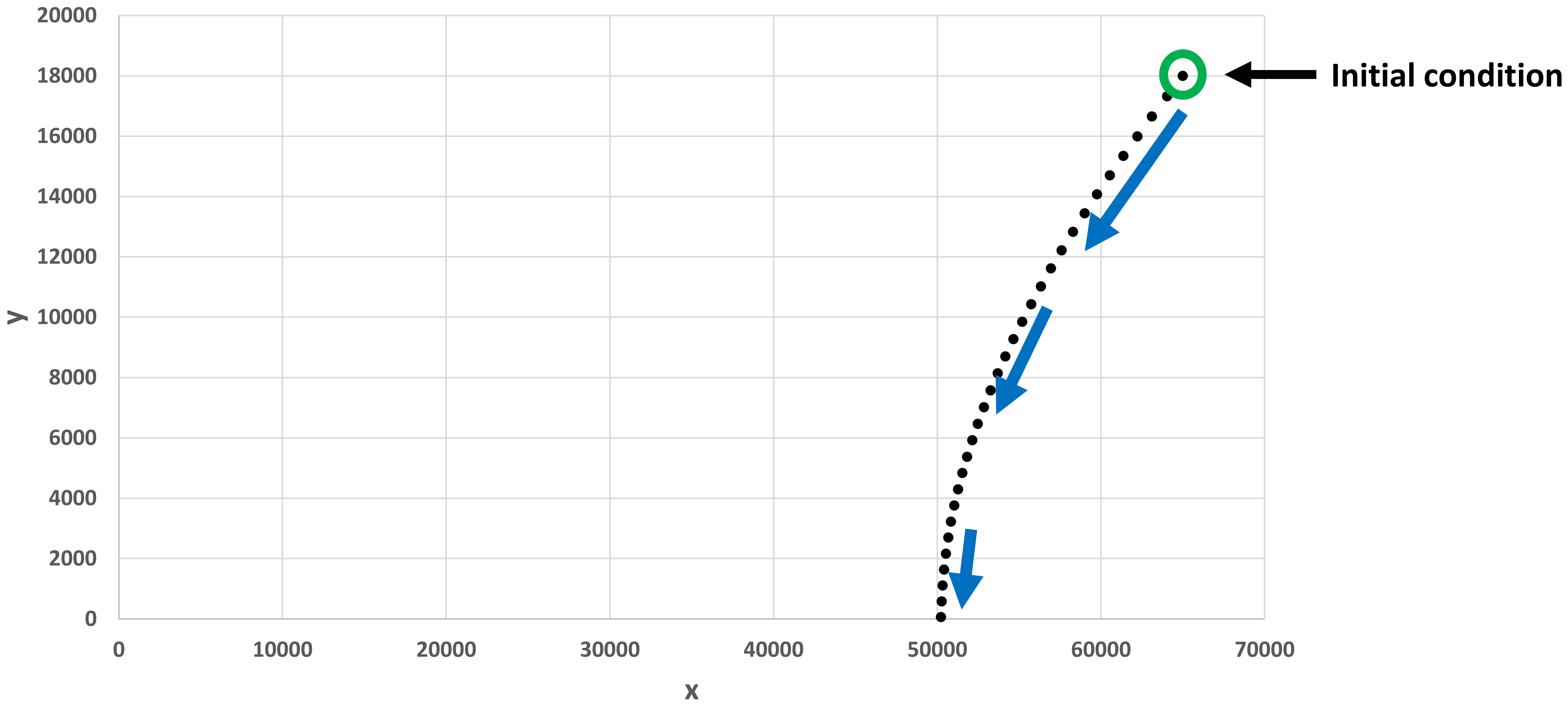
0.05347

$$\begin{cases} \dot{x} = -ay \\ \dot{y} = -bx \end{cases}$$

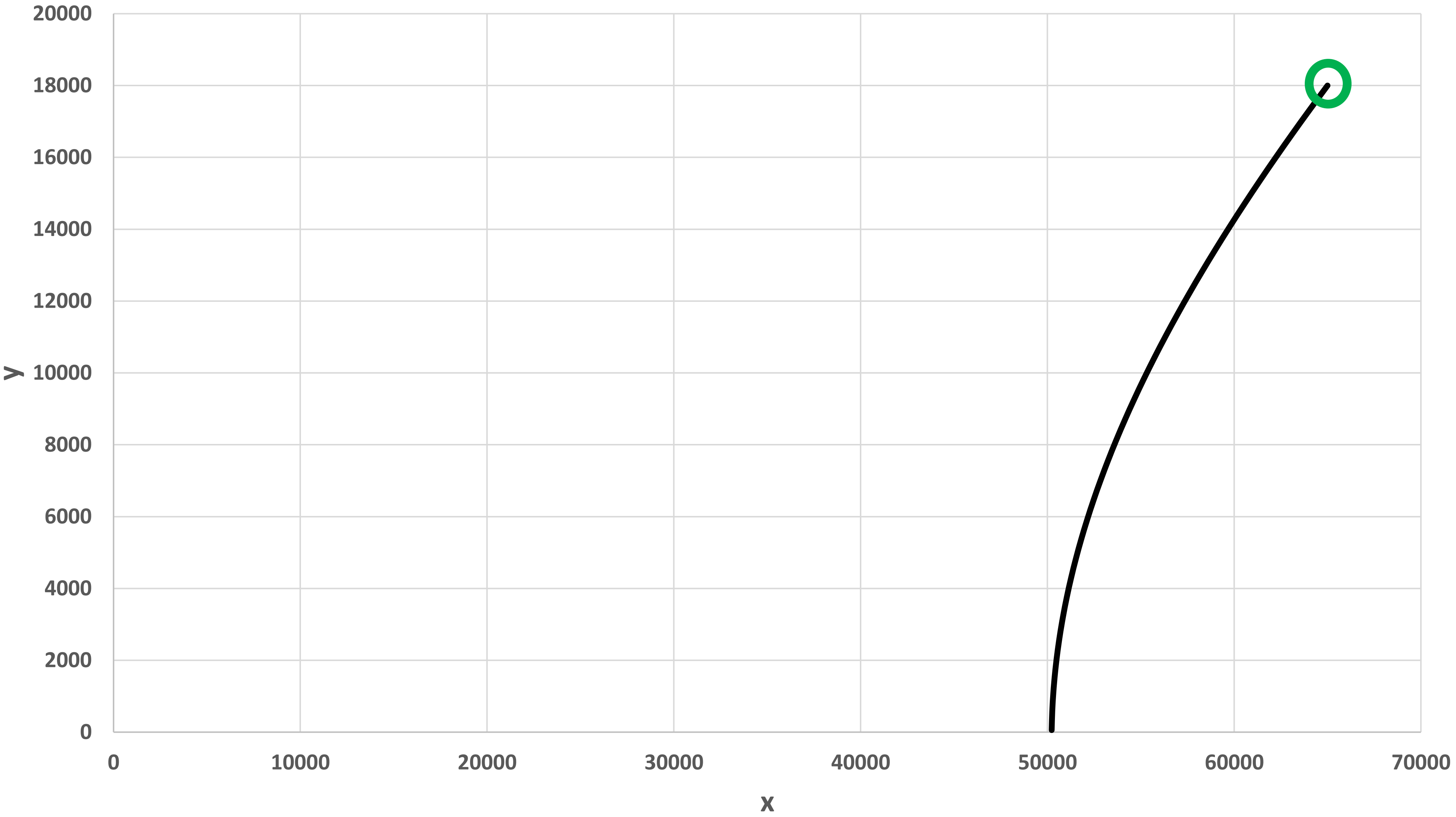
0.01045

Source:

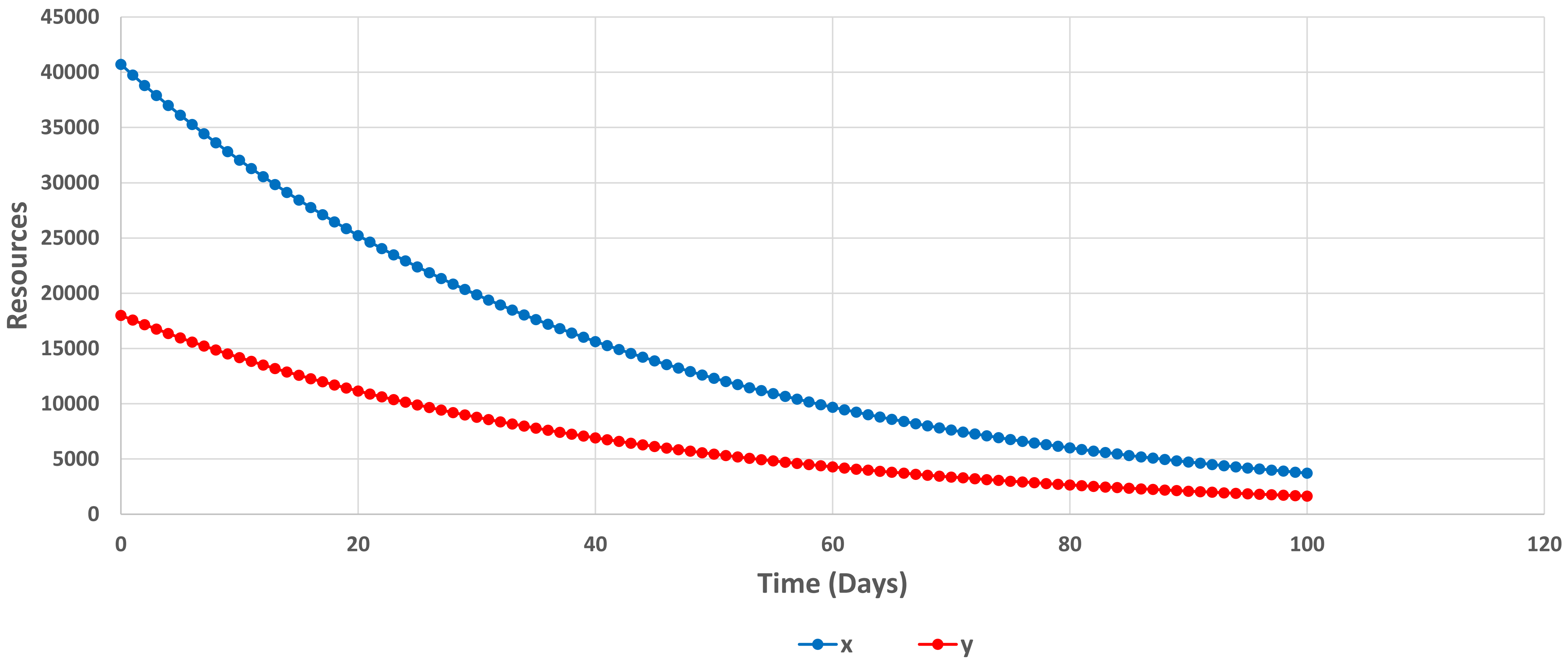
Lohmander, P., Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions,
Journal of Statistics and Computer Science, Vol. 3, Issue 1, 2024, pp. 51-78.
https://www.arfjournals.com/jscs/issue/322
https://www.arfjournals.com/image/catalog/Journals%20Papers/JSCS/2024/No%201%20(2024)/ART_4.pdf

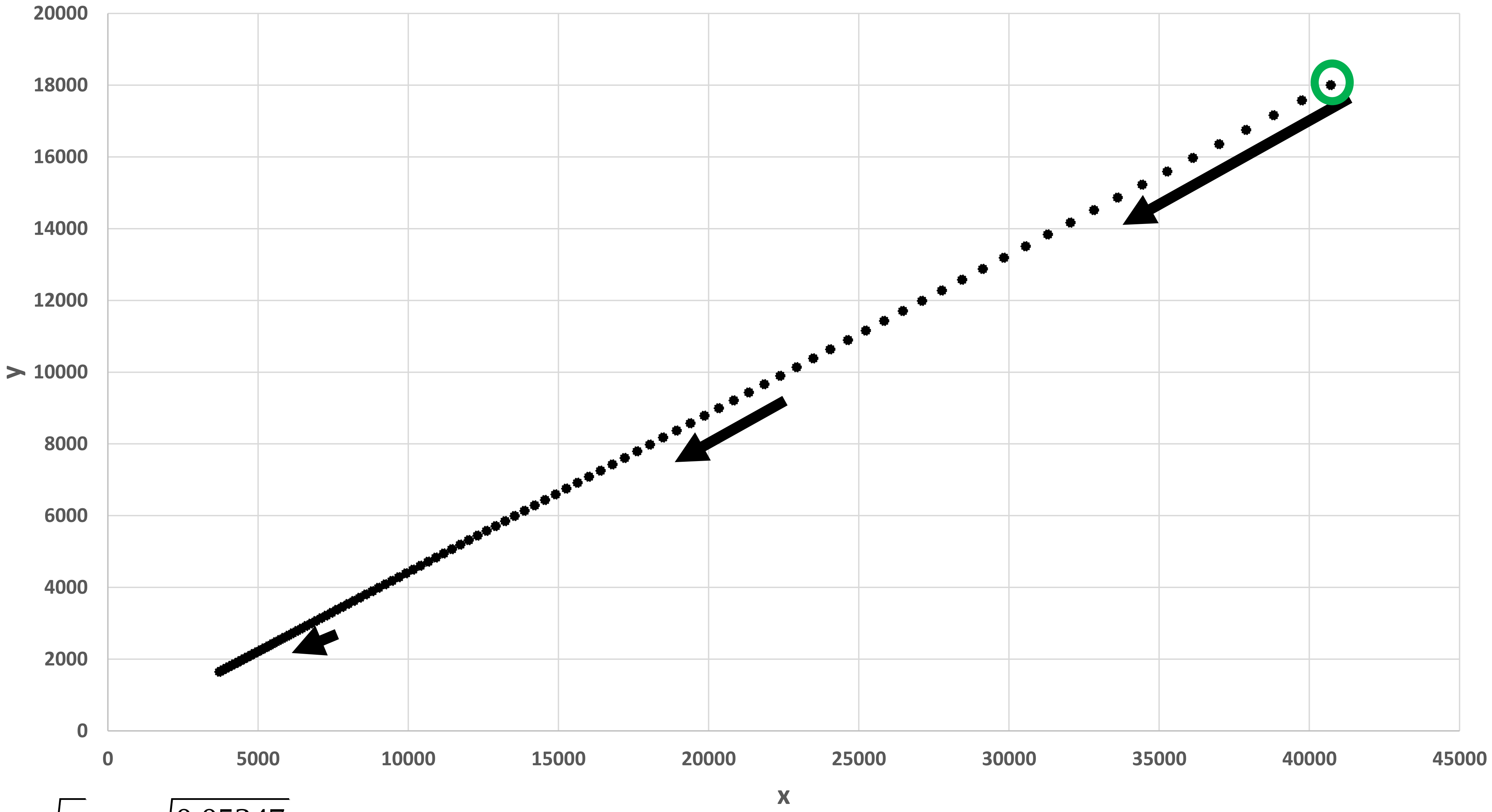


Initial condition
20000
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
y
0
10000
20000
30000
40000
50000
60000
70000
x

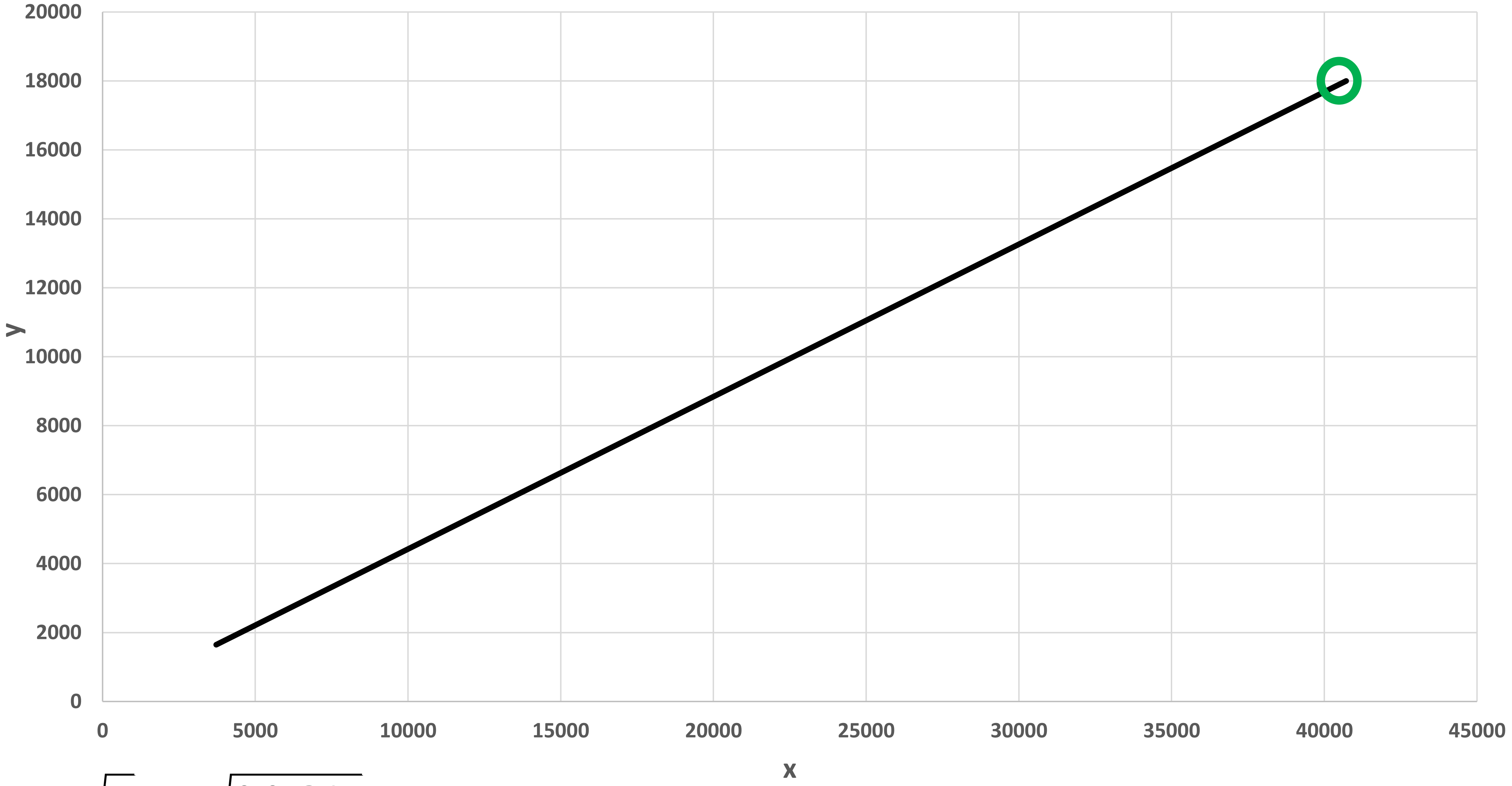

20000
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
y
0
10000
20000
30000
40000
50000
60000
70000
x

$$x_0 = \sqrt{\frac{a}{b}} y_0 = \sqrt{\frac{0.05347}{0.01045}} \times 18000 \approx 40716$$

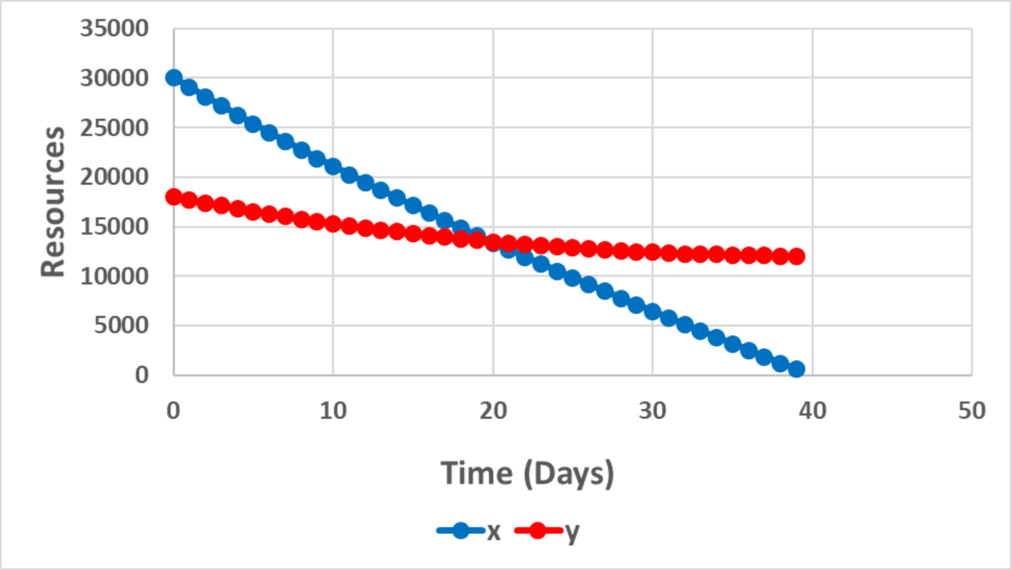




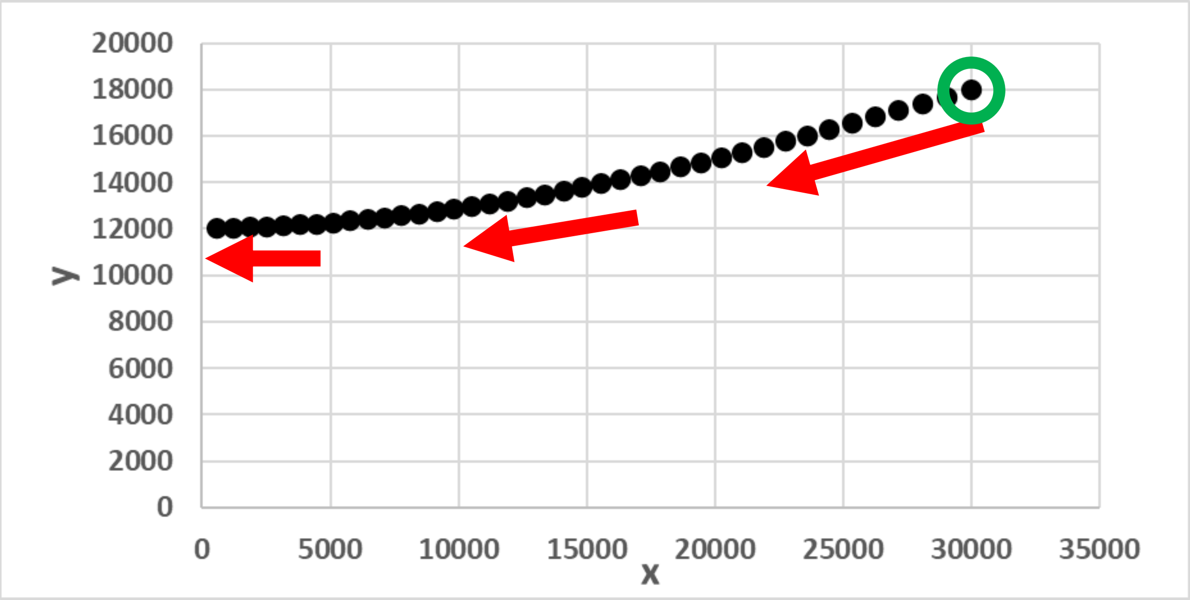
$$x_0 = \sqrt{\frac{a}{b}} y_0 = \sqrt{\frac{0.05347}{0.01045}} \times 18000 \approx 40716$$



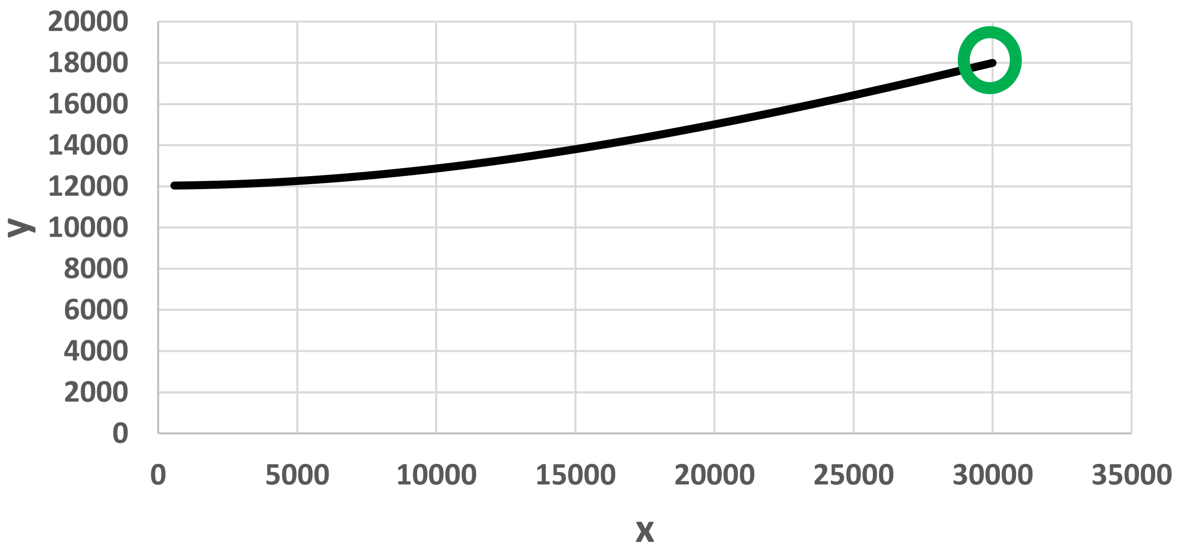
$$x_0 = \sqrt{\frac{a}{b}} y_0 = \sqrt{\frac{0.05347}{0.01045}} \times 18000 \approx 40716$$



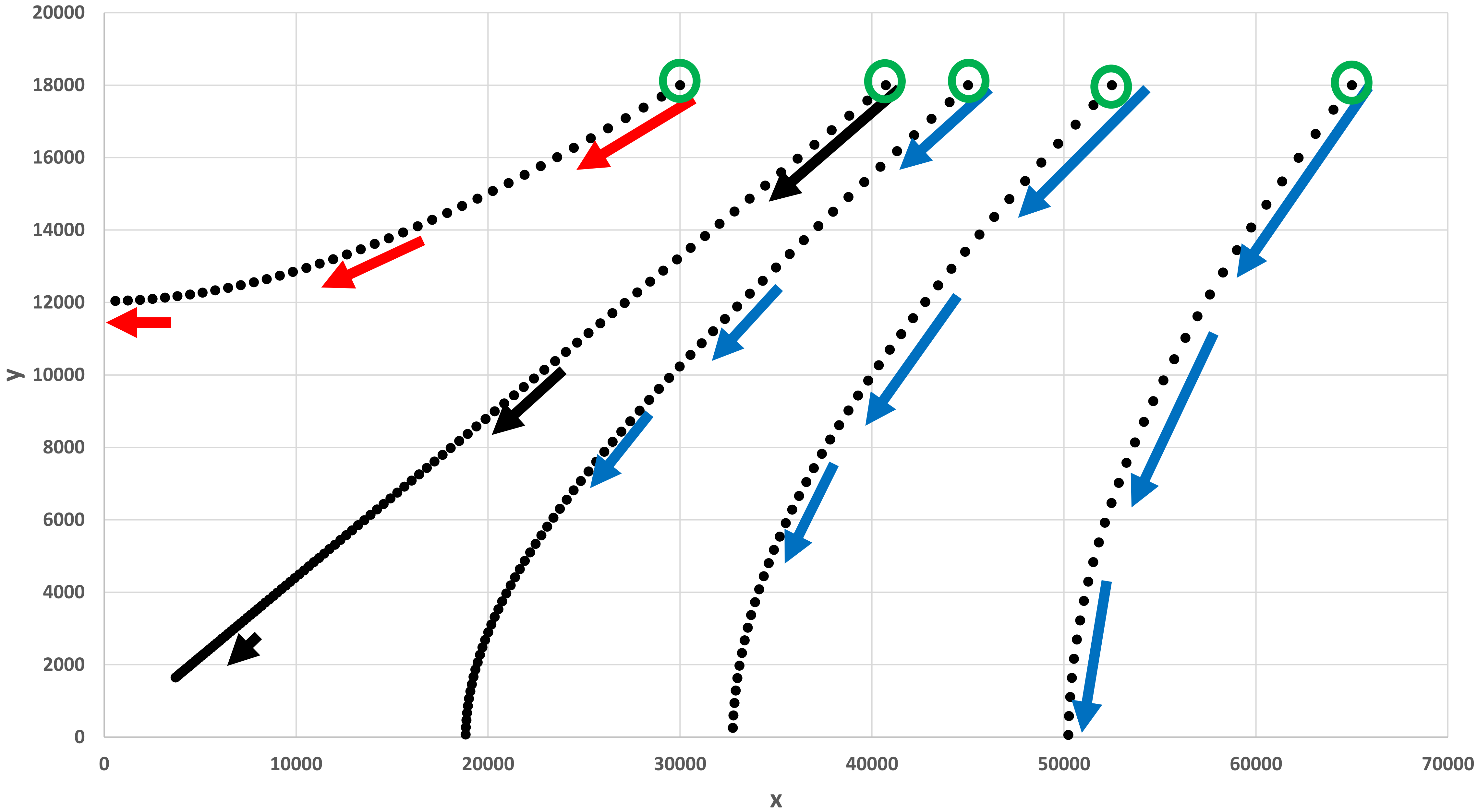
Resources
Time (Days)
x
y



y
x



y
x


20000
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
y
0
10000
20000
30000
40000
50000
60000
70000
x

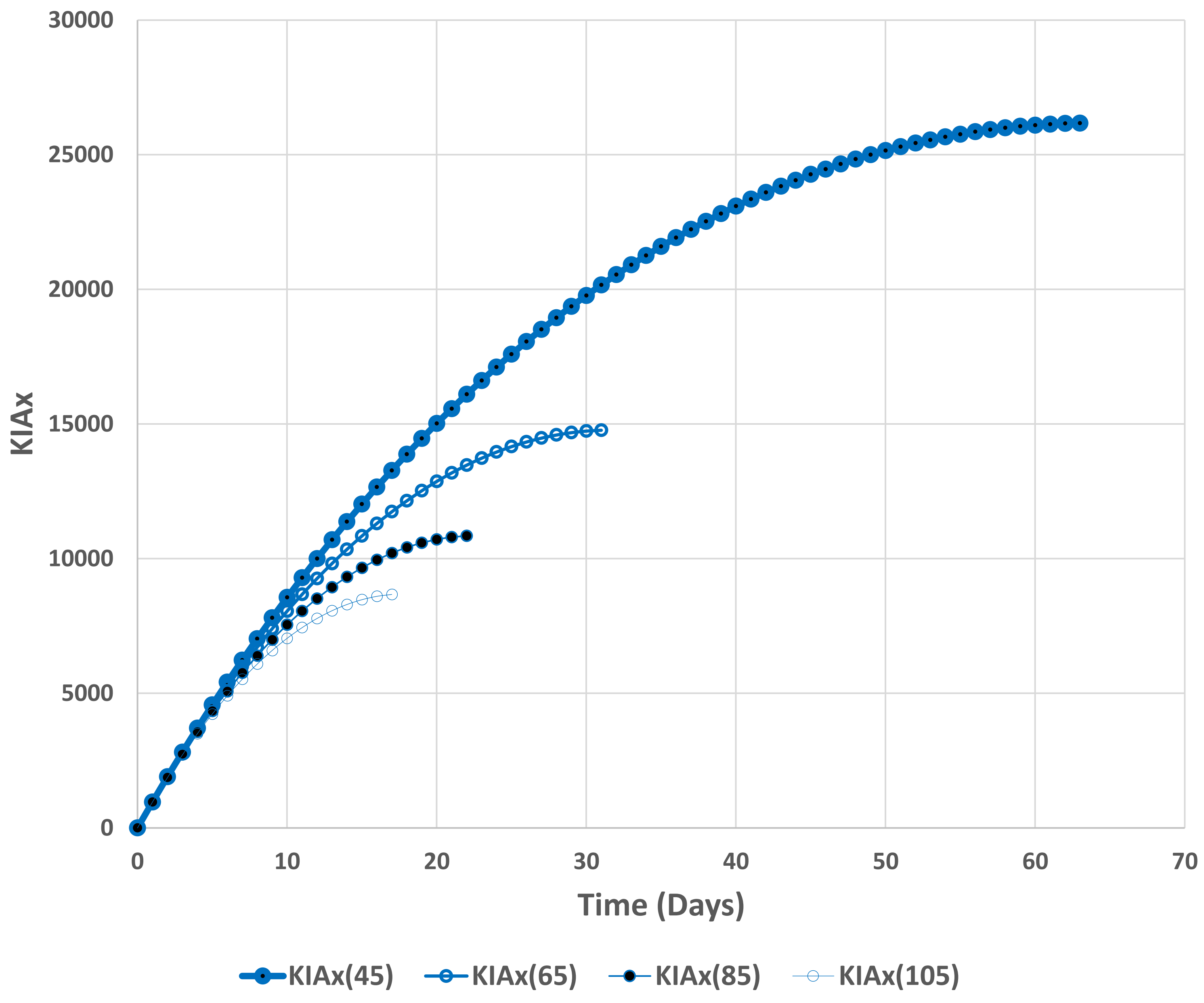


Figure 15.

KIAx denotes the total number of lost x resources, at different points in time, t, until t = T.

T is the point in time, when y(T) = 0. $KIAx(x_0/1000) = x_0 - x(t)$.

a = 0.05347 and b = 0.01045. y_0 = 18000.

In the four different cases, x_0 takes the value 45000, 65000, 85000, or 105000.

The graph is constructed via a discrete time approximation of the differential equation system (1).

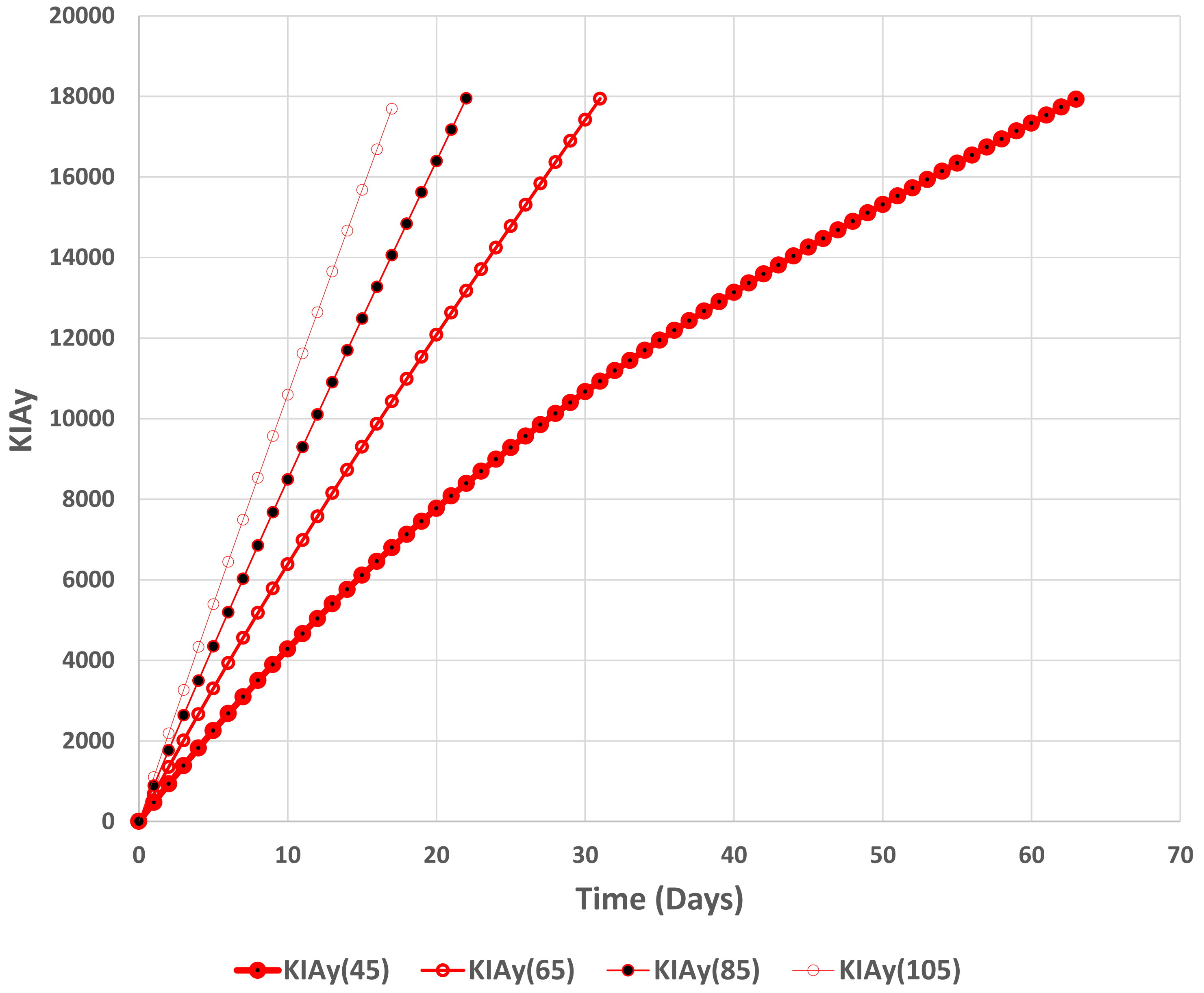


Figure 16.

KIAy is the total number of lost y resources, at different points in time, t, until t = T.

T is the point in time when y(T) = 0. $KIAy(x_0/1000) = y_0 - y(t)$.

a = 0.05347 and b = 0.01045. In all cases, $y_0 = 18000$.

In the four different cases, x_0 takes the value 45000, 65000, 85000, or 105000.

The graph is constructed via a discrete time approximation of the differential equation system (1).

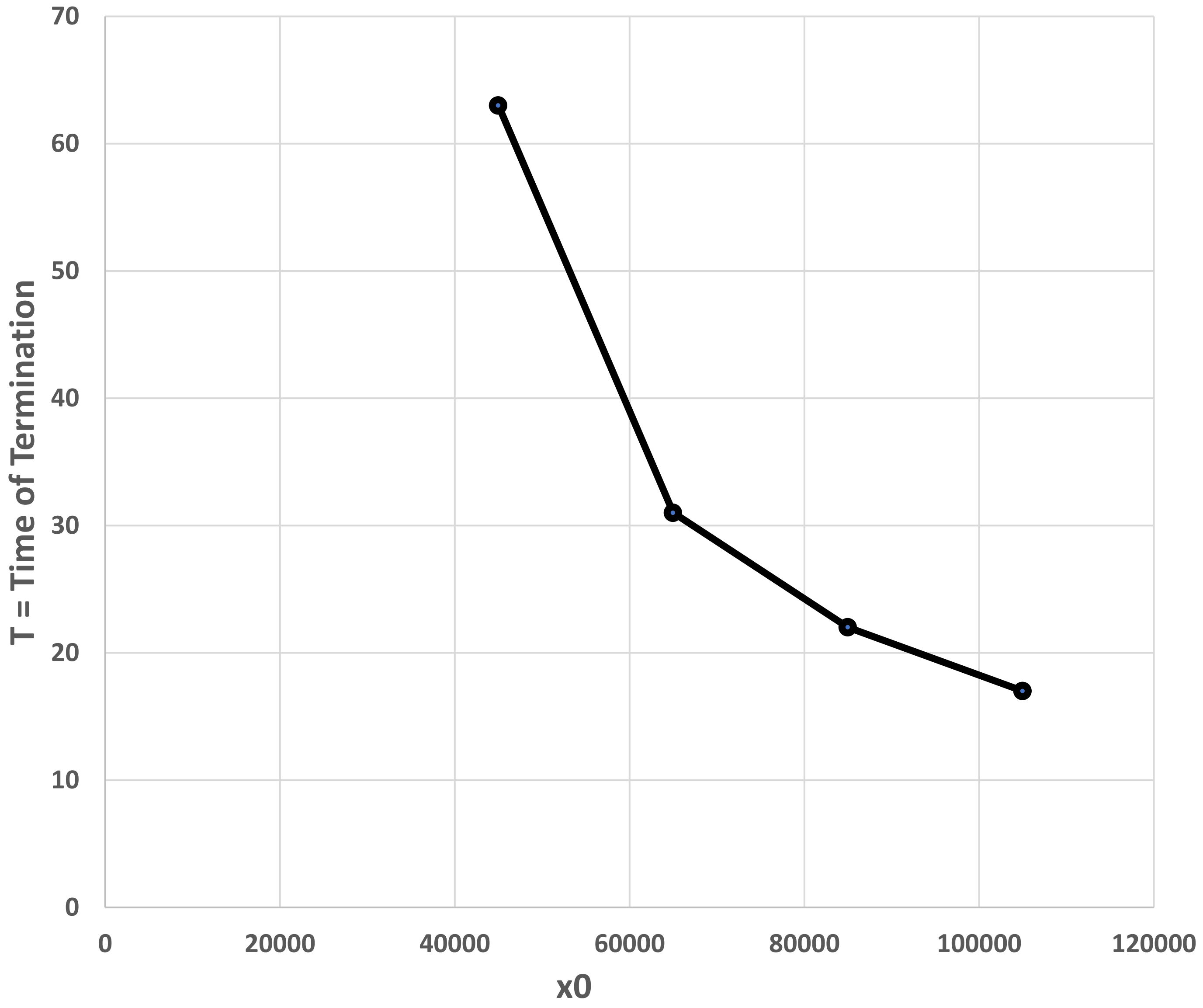


Figure 17.

T, the time of termination, is the point in time, when y(T) = 0.

a = 0.05347 and b = 0.01045.

y_0 = 18000.

In the four different cases, x_0 takes the value 45000, 65000, 85000, or 105000.

The graph is constructed via a discrete time approximation of the differential equation system (1).

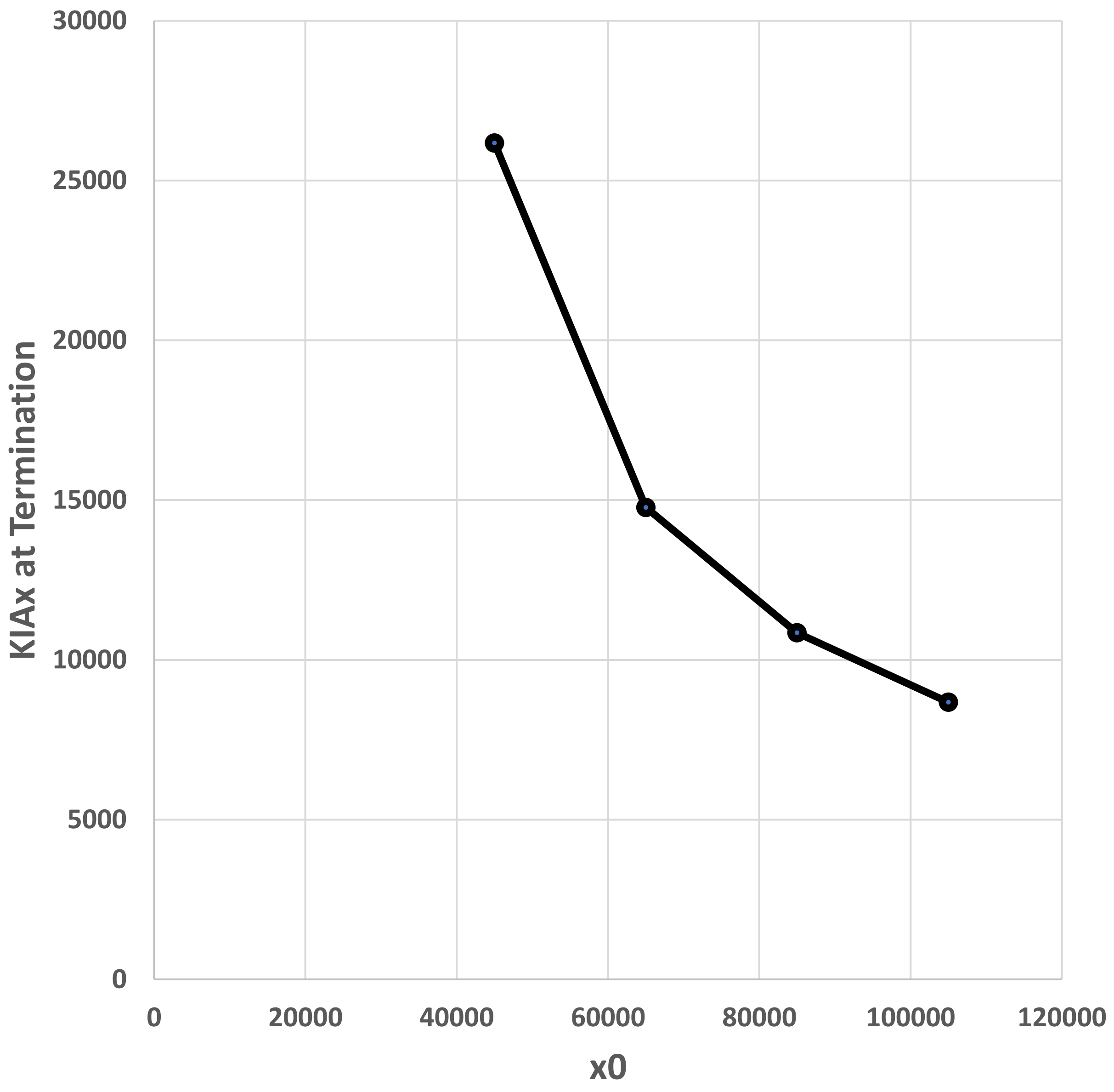


Figure 18.

KIAx at termination is the total number of lost x resources, at time t = T.

T is the point in time, when y(T) = 0.

a = 0.05347 and b = 0.01045.

y_0 = 18000.

In the four different cases, x_0 takes the value 45000, 65000, 85000, or 105000.

The graph is constructed via a discrete time approximation of the differential equation system (1).

Formal analysis:

Briefing on this section:

The complete dynamics of the battle in continuous time is determined. First, the general solution to the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived. This may be interpreted as a 2-dimensional Two Point Boundary Value Problem (TPBVP). Equation (12) corresponds to equation (1), but also includes initial conditions.

We study the differential equation system (12). The state of the system, $(x(t), y(t))$, representing the sizes of the two opposing forces, changes over time, $t, 0 \le t \le T < \infty$. The two parameters, (a,b), are called attrition coefficients. Newtonian notation, with time derivatives marked by dots, is used.

$$\begin{cases} \dot{x} = -ay & (12.a) \\ \dot{y} = -bx & (12.b) \end{cases} \qquad a > 0, b > 0, x(0) = x_0 > 0, y(0) = y_0 > 0 \qquad (12)$$

From (12.a), we get (13).

$$y = -a^{-1}\dot{x} \tag{13}$$

Differentiation of (13) with respect to time, gives (14).

$$\dot{y} = -a^{-1}\ddot{x} \tag{14}$$

(14) and (12.b) give (15). That can be rewritten as (16) and (17), which is a homogenous second order differential equation.

$$-a^{-1}\ddot{x} = -bx \tag{15}$$

$$a^{-1}\ddot{x} - bx = 0 \tag{16}$$

$$\ddot{x} - abx = 0 \tag{17}$$

Let us assume that the functional form (18) is relevant. The parameters (m, λ) are assumed to be strictly different from zero.

$$x(t) = me^{\lambda t}, \quad m \neq 0, \lambda \neq 0, 0 \leq t \leq T < \infty \tag{18}$$

Then, the following procedure can be used to determine the state variable as an explicit function of time. Equations (17) and (18) give (19).

$$\lambda^2 m e^{\lambda t} - abme^{\lambda t} = 0 \tag{19}$$

Equation (19) can be simplified to (20).

$$\left(\lambda^2 - ab\right) m e^{\lambda t} = 0 \tag{20}$$

Equations (18) and (20) imply (21).

$$\lambda^2 - ab = 0 \tag{21}$$

From the quadratic equation (21), we obtain the solution (22).

$$\lambda = \pm\sqrt{ab} \tag{22}$$

$$\lambda = \pm\sqrt{ab} \tag{22}$$

Let r be defined according to (23).

$$r = \sqrt{ab} \tag{23}$$

Clearly, two solutions exist.

$$\lambda_1 = -r \tag{24}$$

$$\lambda_2 = r \tag{25}$$

Observation:

$a > 0 \wedge b > 0$, as we see in equation (12), which means that there are two real roots. These roots have different values. Hence, the general solution of the differential equation is:

$$x(t) = m_1 e^{-rt} + m_2 e^{rt} \tag{26}$$

Furthermore, from (13) we already know that: $y = -a^{-1} \dot{x}$

As a result, we get (27).

$$y(t) = -a^{-1}\left(-rm_1 e^{-rt} + rm_2 e^{rt}\right) \tag{27}$$

The expression (27) may be rewritten as (28).

$$y(t) = \frac{r}{a} m_1 e^{-rt} - \frac{r}{a} m_2 e^{rt} \tag{28}$$

$$
\begin{cases} x(t) = m_1 e^{-rt} + m_2 e^{rt} \\ y(t) = \dfrac{r}{a} m_1 e^{-rt} - \dfrac{r}{a} m_2 e^{rt} \end{cases} \tag{29}
$$

To determine the time path $\left(x(t), y(t)\right)$ we need to know the four parameters $\left(m_1, m_2, a, r\right)$. We already know the initial value of y, $y(0) = y_0$. In this study, we are interested to determine the optimal value of x_0. We want to be sure that we will win the battle, which means that $x(T) > 0$ and $y(T) = 0$ at a point in time, T. This point in time, when the enemy has no more available resource, is denoted the terminal time.

From equation (29), the initial conditions (30) and (31) follow:

$$x(0) = m_1 + m_2 = x_0 \tag{30}$$

$$y(0) = \frac{r}{a} m_1 - \frac{r}{a} m_2 = y_0 \tag{31}$$

The terminal conditions, (32) and (33), are also derived from equation (29):

$$x(T) = m_1 e^{-rT} + m_2 e^{rT} = x_T \tag{32}$$

$$y(T) = \frac{r}{a} m_1 e^{-rT} - \frac{r}{a} m_2 e^{rT} = y_T \tag{33}$$

The nonlinear simultaneous equation system (34) must be satisfied. We assume that a feasible solution exists and that this solution is unique.

$$\begin{cases} m_1 + m_2 = x_0 & (34.a) \\ m_1 e^{-rT} + m_2 e^{rT} = x_T & (34.b) \\ \dfrac{r}{a} m_1 - \dfrac{r}{a} m_2 = y_0 & (34.c) \\ \dfrac{r}{a} m_1 e^{-rT} - \dfrac{r}{a} m_2 e^{rT} = y_T & (34.d) \end{cases} \qquad (34)$$

Determination of $\left(m_1, m_2\right)$:

$$\begin{bmatrix} 1 & 1 \\ s & -s \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{35}$$

$$s = \frac{r}{a} = \frac{\sqrt{ab}}{a} = \sqrt{\frac{b}{a}} \tag{36}$$

$$\left|D\right| = \begin{vmatrix} 1 & 1 \\ s & -s \end{vmatrix} = -2s \tag{37}$$

From Cramer’s rule, we get:

$$m_1 = \frac{\begin{vmatrix} x_0 & 1 \\ y_0 & -s \end{vmatrix}}{|D|} = \frac{-sx_0 - y_0}{-2s} \tag{38}$$

$$m_1 = \frac{x_0 + s^{-1} y_0}{2} = \frac{x_0 + \sqrt{\frac{a}{b}} y_0}{2} \tag{39}$$

$$m_1 = \frac{x_0 + v y_0}{2} > 0 \quad , \quad v = s^{-1} = \sqrt{\frac{a}{b}} \tag{40}$$

$$m_2 = \frac{\begin{vmatrix} 1 & x_0 \\ s & y_0 \end{vmatrix}}{|D|} = \frac{y_0 - sx_0}{-2s} \quad (41)$$

$$m_2 = \frac{x_0 - vy_0}{2} = \frac{x_0 - \sqrt{\frac{a}{b}}y_0}{2} \quad (42)$$

$$m_2 \begin{Bmatrix} > 0 \\ = 0 \\ < 0 \end{Bmatrix} \Leftrightarrow \frac{x_0}{y_0} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \sqrt{\frac{a}{b}} \Leftrightarrow \frac{x_0{}^2}{y_0{}^2} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{a}{b} \Leftrightarrow bx_0{}^2 \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} ay_0{}^2 \quad (43)$$

Observations:

Two different proofs are given in the end of this paper that show that $x(T)=\sqrt{\dfrac{bx_0^{\,2}-ay_0^{\,2}}{b}}$.

If $bx_0^{\,2}>ay_0^{\,2}$, then $y(t)$ reaches zero when $x(t)>0$. In that case, $m_2>0$.

If $bx_0^{\,2}<ay_0^{\,2}$, then $x(t)$ reaches zero when $y(t)>0$. In that case, $m_2<0$.

If $bx_0^{\,2}=ay_0^{\,2}$ (which is extremely unlikely), then $x(t)$ and $y(t)$ both converge to zero. Then, $m_2=0$.

The case when $bx_0^{\,2}=ay_0^{\,2}$ is not further studied in this paper, since the probability of that case is practically zero.

Determination of T.

From now on, we only consider the case where $bx_0^{\ 2} > ay_0^{\ 2}$. Consequently, $y(t)$ reaches zero when $x(t) > 0$ and $m_2 > 0$. Let us determine T as the point in time when $y(T) = y_T = 0$.

$$y_T = sm_1e^{-rT} - sm_2e^{rT} = 0 \tag{44}$$

$$s \times \left(m_1e^{-rT} - m_2e^{rT} \right) = 0$$

$$\neq 0 \qquad\qquad = 0 \tag{45}$$

$$\left(m_1e^{-rT} - m_2e^{rT} \right) = 0 \tag{46}$$

$$e^{-rT}\left(m_1 - m_2 e^{2rT}\right) = 0 \quad (47)$$

$$\neq 0 \qquad = 0$$

$$m_2 e^{2rT} = m_1 \quad (48)$$

$$e^{2rT} = \frac{m_1}{m_2} \quad (49)$$

$$2rT = LN\left(\frac{m_1}{m_2}\right) \quad (50)$$

$$T = \frac{LN\left(\frac{x_0 + vy_0}{x_0 - vy_0}\right)}{2r} = \frac{LN\left(\frac{x_0 + \sqrt{\frac{a}{b}}\,y_0}{x_0 - \sqrt{\frac{a}{b}}\,y_0}\right)}{2\sqrt{ab}} \quad (51)$$

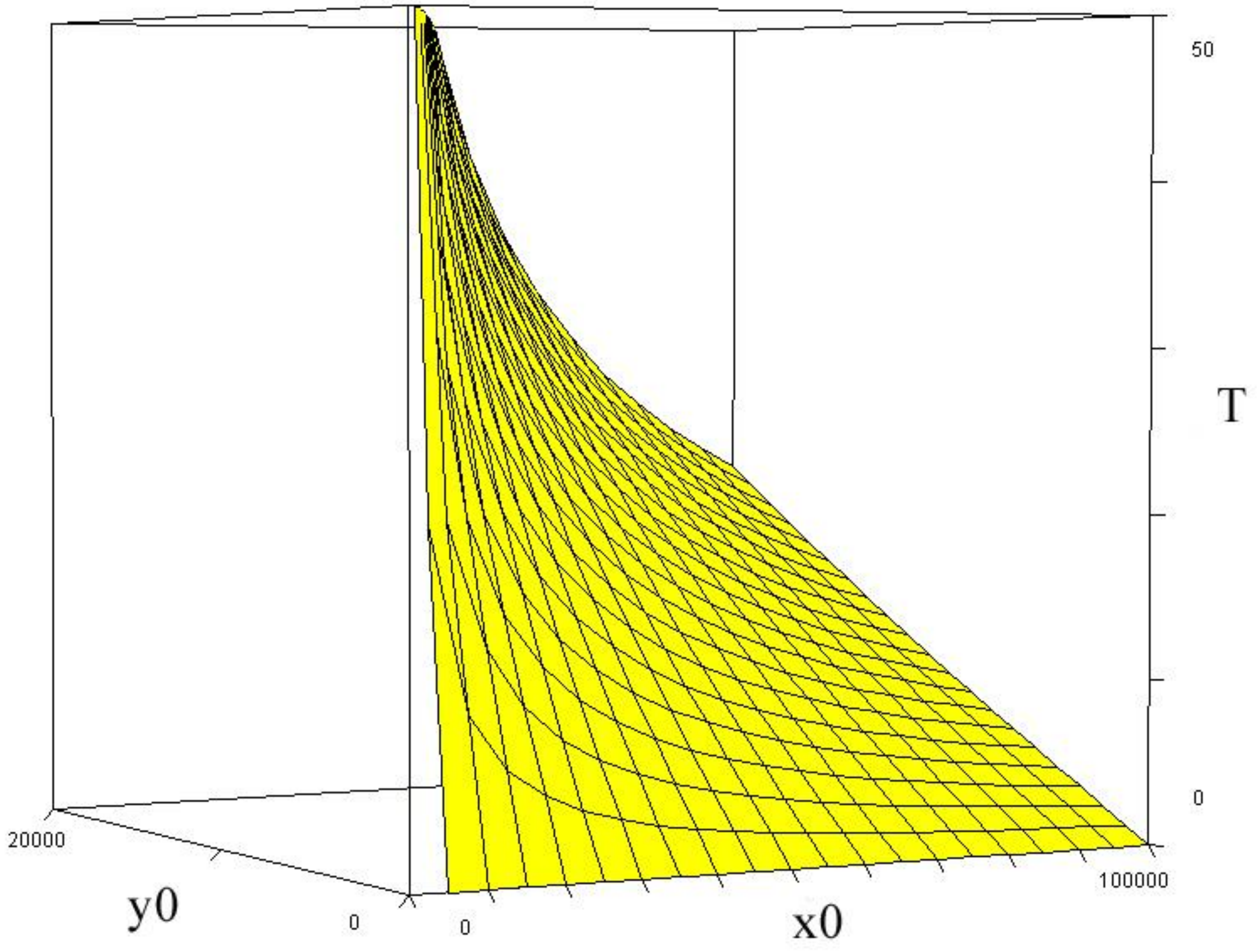


Figure 15.

T(x0, y0).

a = 0.05347, b = 0.01045.

Compare equation (51).

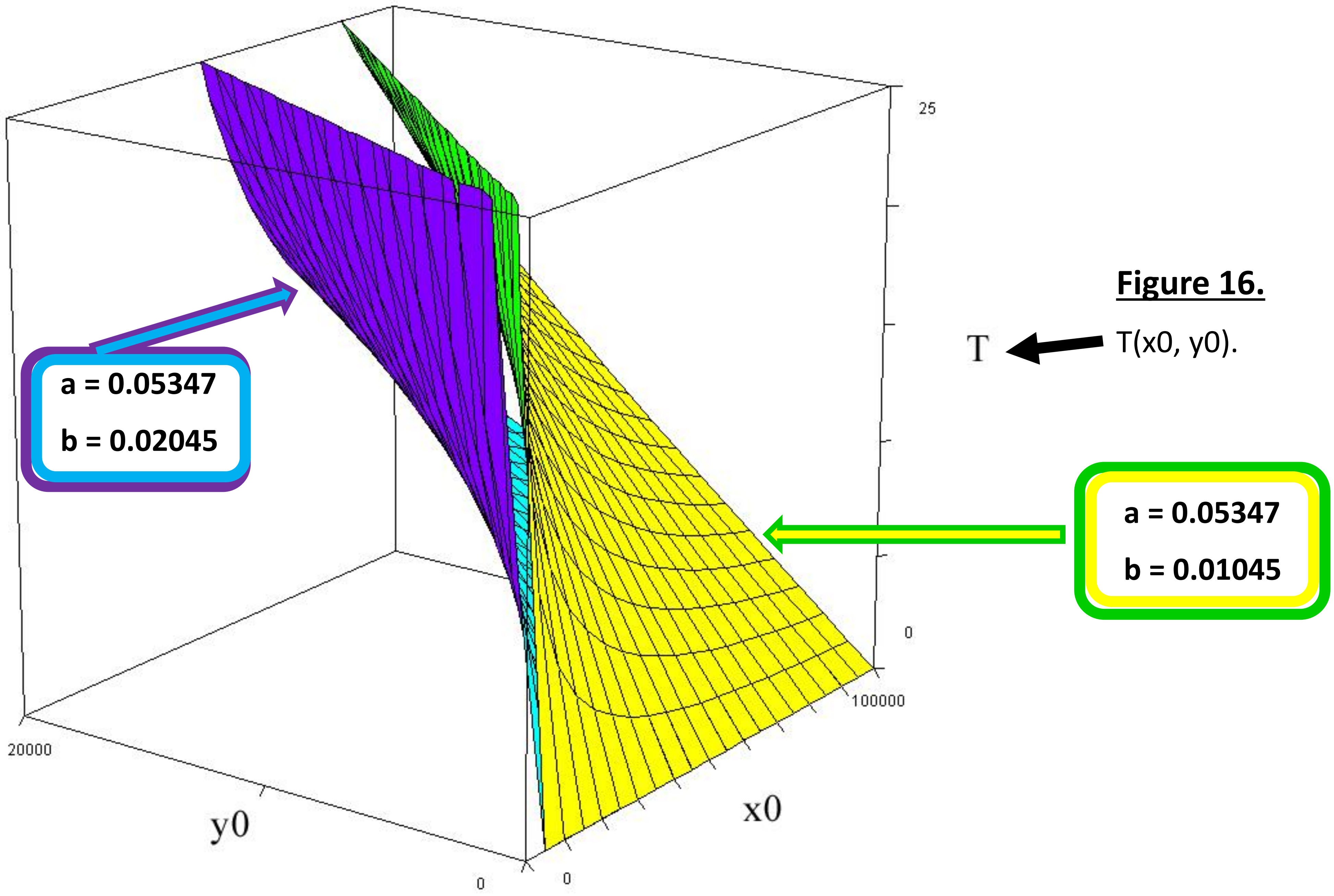


Figure 16.

T(x0, y0).

Determination of the derivative of T with respect to x_0.

$$\frac{dT}{dx_0} = (2r)^{-1}\left(\frac{x_0 - vy_0}{x_0 + vy_0}\right)\left(\frac{1\times(x_0 - vy_0) - (x_0 + vy_0)\times 1}{(x_0 - vy_0)^2}\right) \qquad (52)$$

$$\frac{dT}{dx_0} = (2r)^{-1}\frac{-2vy_0}{(x_0 + vy_0)(x_0 - vy_0)} \qquad (53)$$

$$\frac{dT}{dx_0} = \frac{-vy_0}{r\left(x_0 + vy_0\right)\left(x_0 - vy_0\right)} \quad (54)$$

$$\frac{dT}{dx_0} = \frac{-vy_0}{r\left({x_0}^2 - v^2{y_0}^2\right)} \quad (55)$$

$$\frac{dT}{dx_0} = \frac{-\sqrt{\frac{a}{b}}y_0}{\sqrt{ab}\left({x_0}^2 - \frac{a}{b}{y_0}^2\right)} \quad (56)$$

$$\frac{dT}{dx_0} = \frac{-y_0}{b\left({x_0}^2 - \frac{a}{b}{y_0}^2\right)} \quad (57)$$

$$\frac{dT}{dx_0} = \frac{-y_0}{b{x_0}^2 - a{y_0}^2} < 0 \quad (58)$$

Determination of the second derivative of T with respect to x_0.

$$\frac{d^2T}{d{x_0}^2} = \frac{-(-y_0)\,2bx_0}{\left(b{x_0}^2 - a{y_0}^2\right)^2} \tag{59}$$

$$\frac{d^2T}{d{x_0}^2} = \frac{2bx_0y_0}{\left(b{x_0}^2 - a{y_0}^2\right)^2} > 0 \tag{60}$$

Determination of x_T via the function x(t) and the value of T when $y_T = 0$:

$$x(T) = m_1 e^{-rT} + m_2 e^{rT} = x_T \quad (61)$$

$$x(T) = \left(\frac{x_0 + v y_0}{2} \right) e^{-rT} + \left(\frac{x_0 - v y_0}{2} \right) e^{rT} \quad (62)$$

$$x(T)=\left(\frac{x_0+vy_0}{2}\right)e^{-r\left(\frac{LN\left(\frac{x_0+vy_0}{x_0-vy_0}\right)}{2r}\right)}+\left(\frac{x_0-vy_0}{2}\right)e^{r\left(\frac{LN\left(\frac{x_0+vy_0}{x_0-vy_0}\right)}{2r}\right)} \quad (63)$$

$$x(T)=\left(\frac{x_0+vy_0}{2}\right)e^{-\left(\frac{LN\left(\frac{x_0+vy_0}{x_0-vy_0}\right)}{2}\right)}+\left(\frac{x_0-vy_0}{2}\right)e^{\left(\frac{LN\left(\frac{x_0+vy_0}{x_0-vy_0}\right)}{2}\right)} \quad (64)$$

$$x(T)=\left(\frac{x_0+vy_0}{2}\right)\sqrt{\frac{x_0-vy_0}{x_0+vy_0}}+\left(\frac{x_0-vy_0}{2}\right)\sqrt{\frac{x_0+vy_0}{x_0-vy_0}} \quad (65)$$

$$x(T)=\frac{\sqrt{x_0+vy_0}\sqrt{x_0-vy_0}}{2}+\frac{\sqrt{x_0-vy_0}\sqrt{x_0+vy_0}}{2} \quad (66)$$

$$x(T)=\sqrt{x_0+vy_0}\sqrt{x_0-vy_0} \quad (67)$$

$$\left(x(T)\right)^2=\left(x_0+vy_0\right)\left(x_0-vy_0\right) \quad (68)$$

$$\left(x(T)\right)^2 = x_0{}^2 - v^2 y_0{}^2 \qquad (69)$$

$$x(T) = \sqrt{x_0{}^2 - v^2 y_0{}^2} \qquad (70)$$

$$x(T) = \sqrt{x_0{}^2 - \left(\frac{a}{b}\right) y_0{}^2} \qquad (71)$$

$$x(T) = \sqrt{\frac{b x_0{}^2 - a y_0{}^2}{b}} \qquad (72)$$

Alternative method to determine x_T:

$$\begin{cases} \dfrac{dx}{dt} = -ay \\ \dfrac{dy}{dt} = -bx \end{cases} \qquad (73)$$

$$\frac{dx}{dy} = \frac{-ay}{-bx} \qquad (74)$$

$$bx\,dx = ay\,dy \qquad (75)$$

$$\int_{x_0}^{x_T} bx\,dx = \int_{y_0}^{y_T} ay\,dy \qquad (76)$$

$$b\left[\frac{x^2}{2}\right]_{x_0}^{x_T} = a\left[\frac{y^2}{2}\right]_{y_0}^{y_T} \tag{77}$$

$$b\left(\frac{x_T{}^2}{2} - \frac{x_0{}^2}{2}\right) = a\left(\frac{y_T{}^2}{2} - \frac{y_0{}^2}{2}\right) \tag{78}$$

$$b\left(x_T{}^2 - x_0{}^2\right) = a\left(y_T{}^2 - y_0{}^2\right) \tag{79}$$

$$b\left(x_T{}^2 - x_0{}^2\right) = a\left(-y_0{}^2\right) \quad , \quad y_T = 0 \tag{80}$$

$$b{x_T}^2 = b{x_0}^2 - a{y_0}^2 \quad , \quad y_T = 0 \tag{81}$$

$${x_T}^2 = \frac{b{x_0}^2 - a{y_0}^2}{b} \quad , \quad y_T = 0 \tag{82}$$

$$x_T = \sqrt{\frac{b{x_0}^2 - a{y_0}^2}{b}} \quad , \quad y_T = 0 \tag{83}$$

Q.E.D.

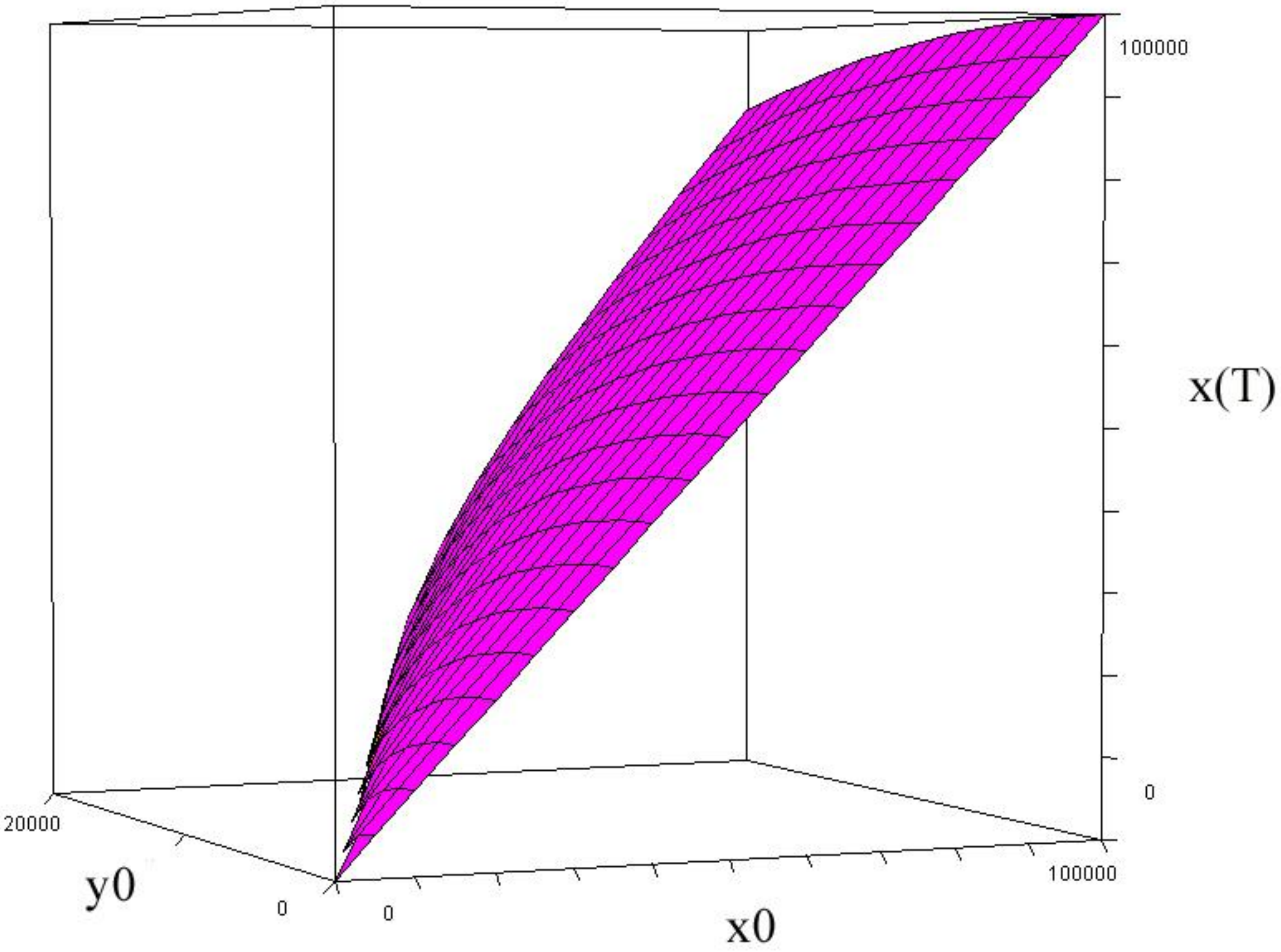


Figure 17.

x_T(x0,y0).

a = 0.05347, b = 0.01045.

Compare equation (83).

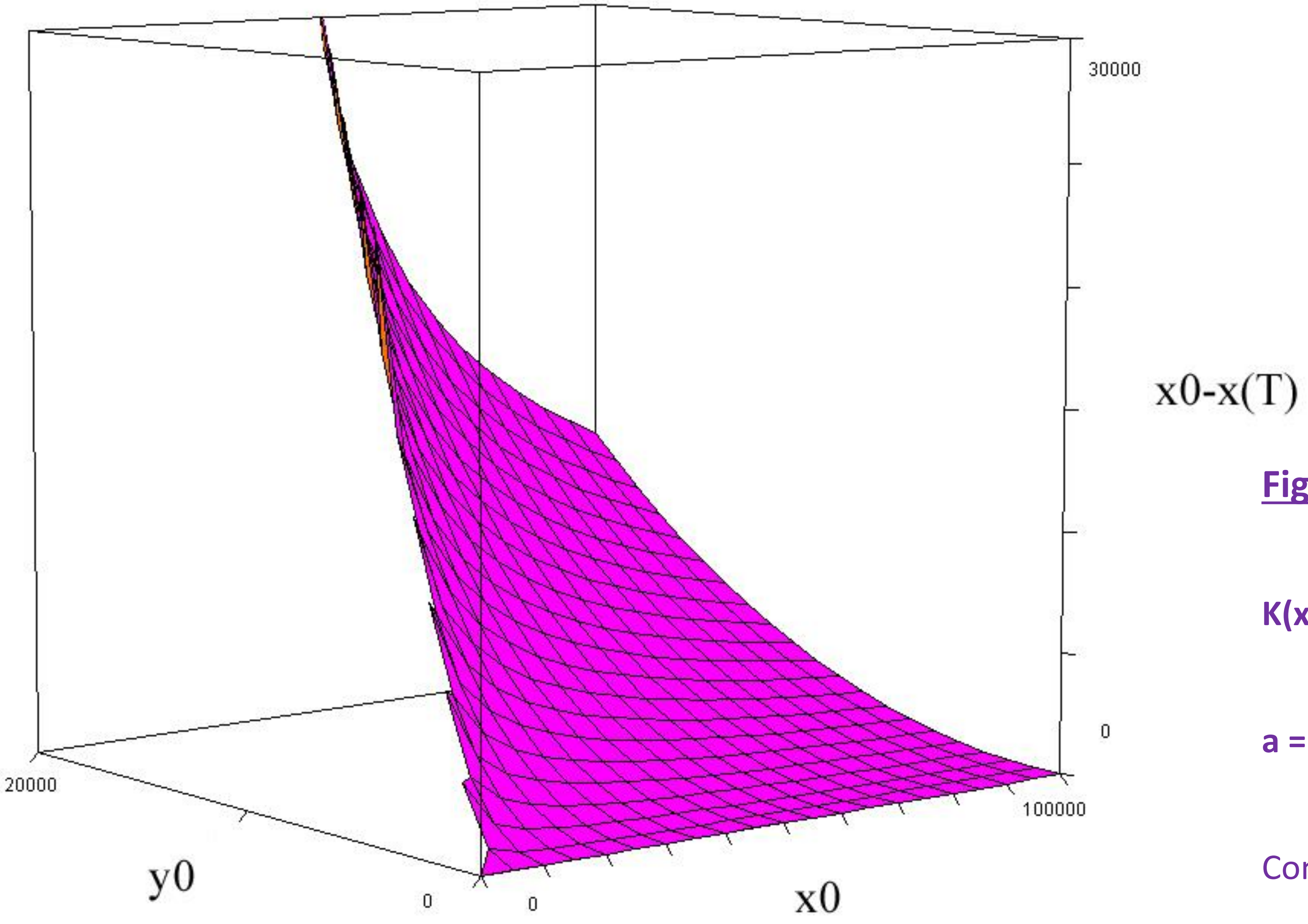


Figure 18.

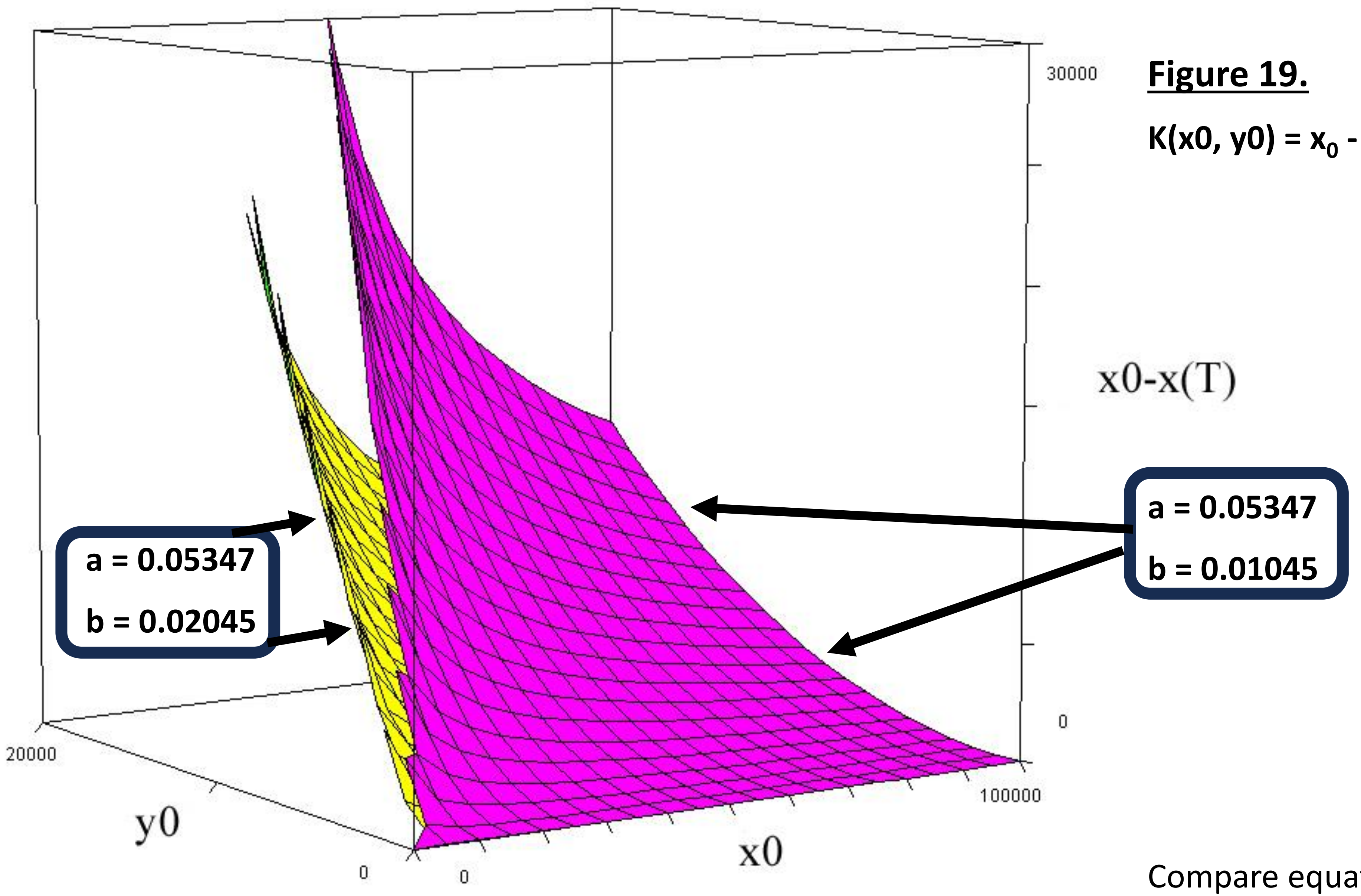
$K(x0, y0) = x_0 - x_T(x0,y0)$.

a = 0.05347, b = 0.01045.

Compare equation (83).

Figure 19.

$K(x0, y0) = x_0 - x_T(x0,y0)$.



Compare equation (83).

Determination of the derivative of x_T with respect to x_0 when $y_T = 0$:

$$x_T = \sqrt{\frac{b{x_0}^2 - a{y_0}^2}{b}} \quad , \quad y_T = 0 \tag{84}$$

$$x_T = b^{-\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{\frac{1}{2}} \quad , \quad y_T = 0 \tag{85}$$

$$\frac{dx_T}{dx_0} = b^{-\frac{1}{2}}\left(\frac{1}{2}\right)\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\left(2bx_0\right) \tag{86}$$

$$\frac{dx_T}{dx_0} = b^{\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}} x_0 > 0 \tag{87}$$

$$\frac{dx_T}{dx_0} = \frac{\sqrt{b}\, x_0}{\sqrt{bx_0{}^2 - ay_0{}^2}} > 0 \tag{88}$$

Determination of the second derivative of x_T with respect to x_0 when $y_T = 0$:

$$\frac{d^2 x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(-\frac{1}{2}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{3}{2}} 2bx_0{}^2 + \left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\right) \tag{89}$$

$$\frac{d^2 x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(-\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{3}{2}} bx_0{}^2 + \left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\right) \tag{90}$$

$$\frac{d^2 x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\left(-\left(bx_0{}^2 - ay_0{}^2\right)^{-1} bx_0{}^2 + 1\right) \tag{91}$$

$$\frac{d^2x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\left(\frac{-bx_0{}^2}{bx_0{}^2 - ay_0{}^2} + 1\right) \tag{92}$$

$$\frac{d^2x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\left(\frac{1}{bx_0{}^2 - ay_0{}^2}\right)\left(-bx_0{}^2 + bx_0{}^2 - ay_0{}^2\right) \tag{93}$$

$$\frac{d^2x_T}{dx_0{}^2} = b^{\frac{1}{2}}\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{3}{2}}\left(-ay_0{}^2\right) \tag{94}$$

$$\frac{d^2x_T}{dx_0{}^2} = \frac{-a\sqrt{b}\,y_0{}^2}{\left(bx_0{}^2 - ay_0{}^2\right)^{\frac{3}{2}}} < 0 \tag{95}$$

Summary of important results

$$T = \frac{LN\left(\dfrac{x_0 + \sqrt{\dfrac{a}{b}}\, y_0}{x_0 - \sqrt{\dfrac{a}{b}}\, y_0}\right)}{2\sqrt{ab}} \qquad (96)$$

$$\frac{dT}{dx_0} = \frac{-y_0}{b{x_0}^2 - a{y_0}^2} < 0 \qquad (97)$$

$$\frac{d^2T}{d{x_0}^2} = \frac{2bx_0y_0}{\left(b{x_0}^2 - a{y_0}^2\right)^2} > 0 \qquad (98)$$

$$x_T = \sqrt{\frac{b{x_0}^2 - a{y_0}^2}{b}} \quad , \quad y_T = 0 \tag{99}$$

$$\frac{dx_T}{dx_0} = \frac{\sqrt{b}\, x_0}{\sqrt{b{x_0}^2 - a{y_0}^2}} > 0 \tag{100}$$

$$\frac{d^2 x_T}{d{x_0}^2} = \frac{-a\sqrt{b}\, {y_0}^2}{\left(b{x_0}^2 - a{y_0}^2\right)^{\frac{3}{2}}} < 0 \tag{101}$$

Economic optimization in the deterministic case:

$$\max_{x_0} \pi\left(x_0; a, b, c_T, c_{x_T}, G, y_0\right)$$

- c_T: Cost per time unit of delay
- c_{x_T}: Cost per unit of lost resources
- G: Revenue from instant access to free territory

$$\max_{x_0} \pi = \overbrace{-C\left(x_0\right)}^{\text{Total cost of deployment}} + \overbrace{G}^{\text{Revenue from territory}} \overbrace{- c_T T\left(x_0, y_0, a, b\right)}^{\text{Cost of delay}} \overbrace{- c_{x_T}\left(x_0 - x_T\left(x_0, y_0, a, b\right)\right)}^{\text{Cost of lost units}}$$

$$\max_{x_0} \pi = -C\left(x_0\right) + G - c_T \frac{LN\left(\dfrac{x_0 + \sqrt{\dfrac{a}{b}}\, y_0}{x_0 - \sqrt{\dfrac{a}{b}}\, y_0}\right)}{2r} - c_{x_T}\left(x_0 - \sqrt{\frac{b x_0{}^2 - a y_0{}^2}{b}}\right)$$

The Figures 20 and 21 illustrate the objective function (104) as a function of the initial sizes of the two forces.

The functions and values in Figure 20 are:

$C(x_0) = 1000 + 1x_0$

$G = 200000$

$c_T = 730$

$c_{xT} = 2$

$a = 0.05347$

$b = 0.01045$

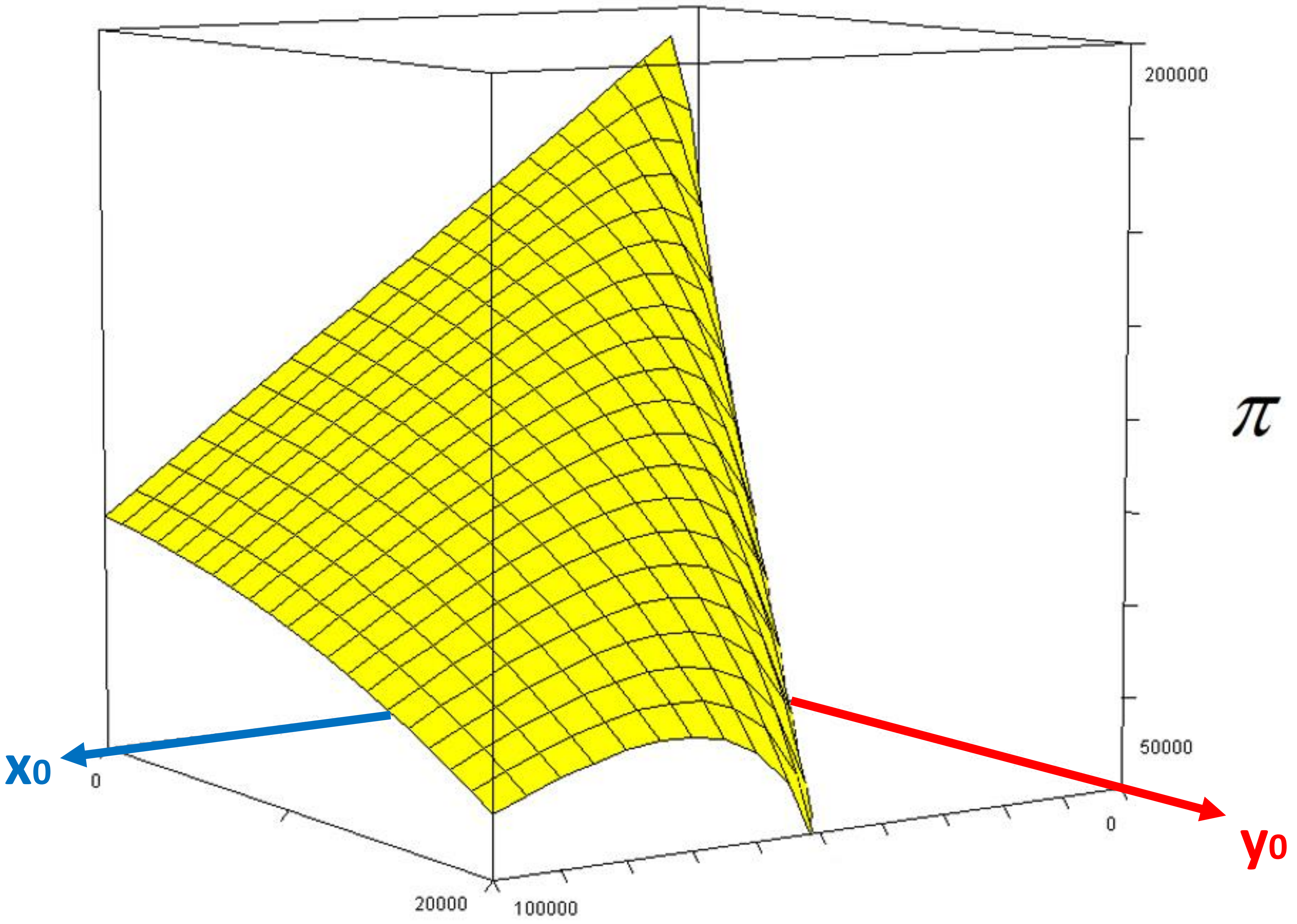
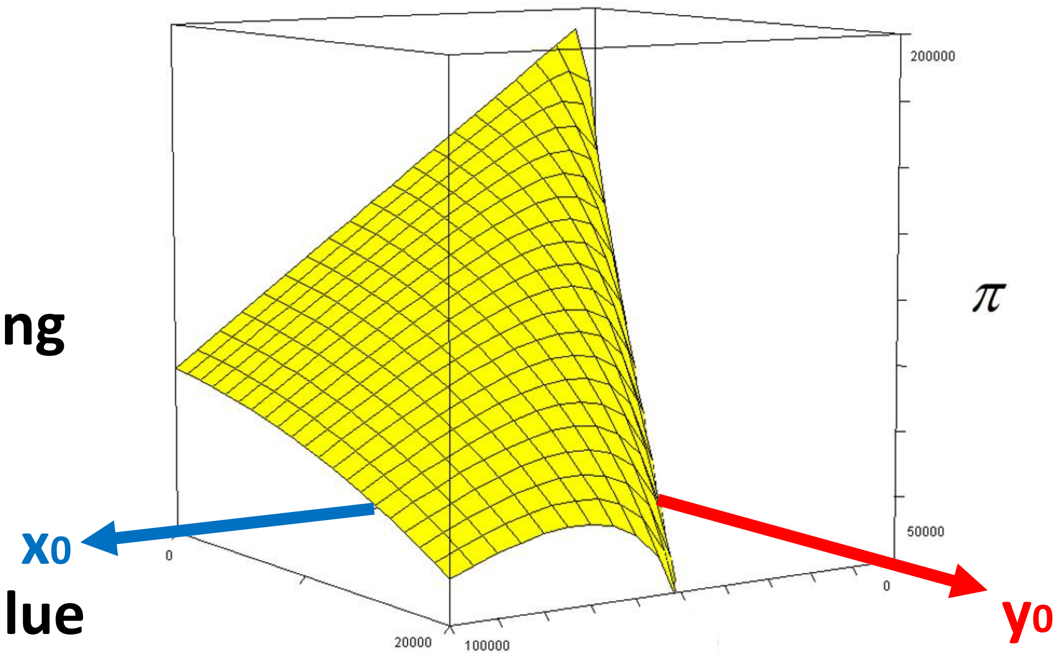


Figure 20.

The objective function in equation (104), as a function of the initial sizes of the two forces.

The graph illustrates that the optimal value of x_0 is an increasing function of y_0.



Furthermore, the optimal value of the objective function of the commander of force x, is a decreasing function of the initial size of the force y.

Clearly, if the value y_0 would have a much larger value than 20000, as illustrated in the graph, the maximum of the objective function value, would be strictly negative.

Then, the optimal decision of the commander of the x forces would be not to participate in the battle at all.

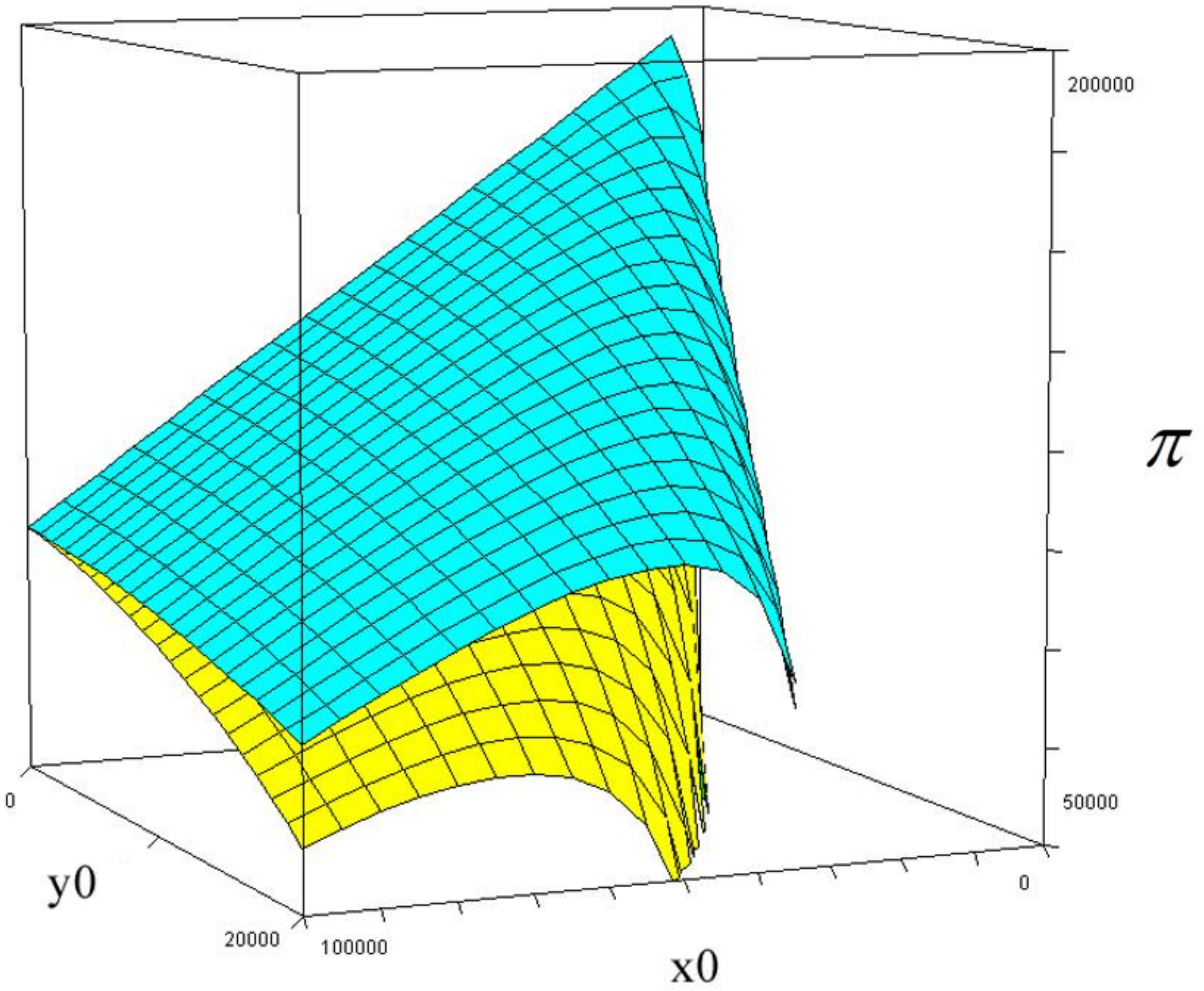


Figure 21.

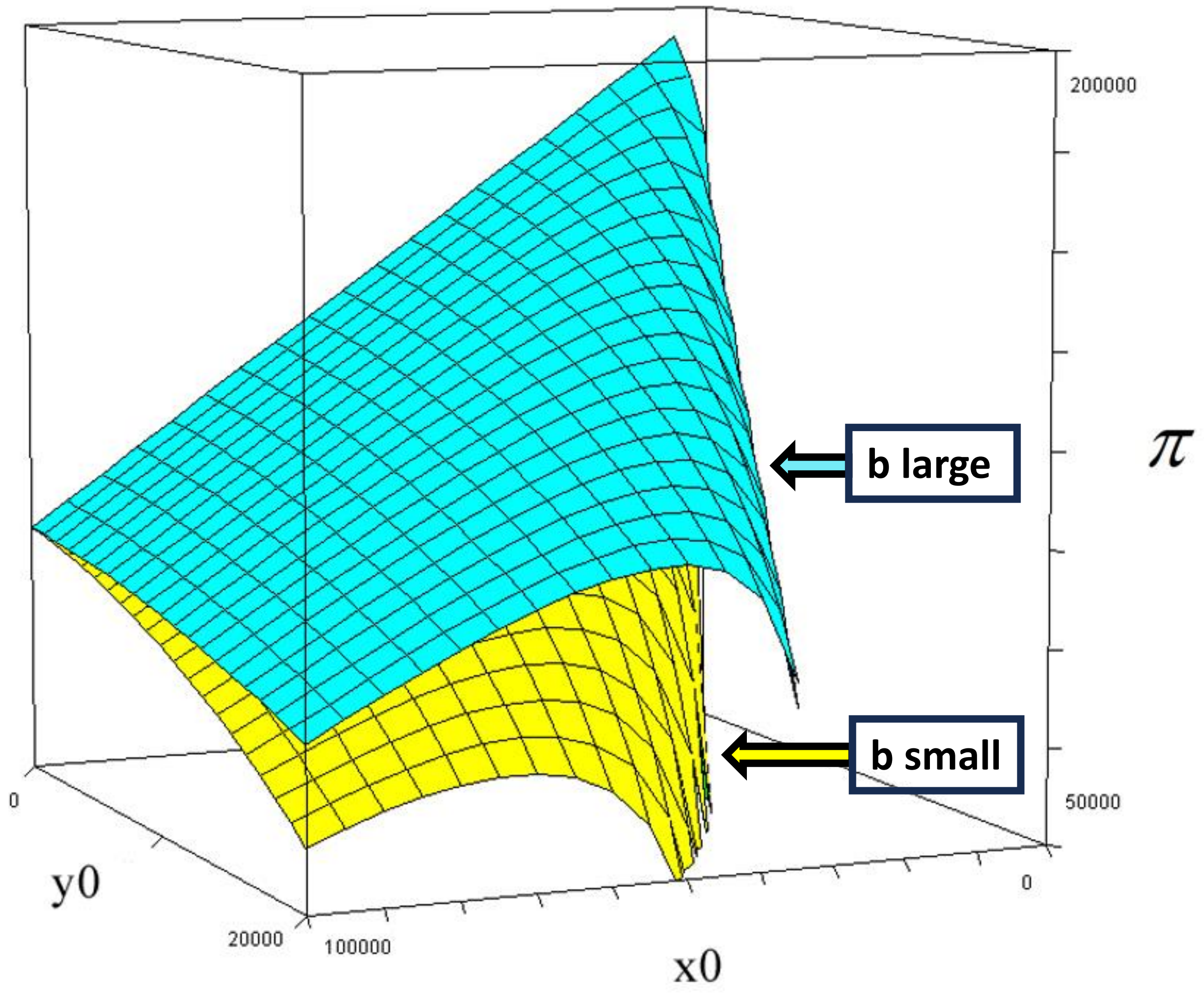
The objective function in equation (104), as a function of the initial sizes of the two forces, with alternative values of the attrition coefficient “b”.

Yellow:

a = 0.05347, b = 0.01045

Turquoise:

a = 0.05347, b = 0.02045



The graph illustrates that the objective function value of the commander of the x forces is an increasing function of the attrition coefficient b,

and that the optimal number of units x to send to the battle field is a decreasing function of b,

for all possible sizes of the enemy force, if the optimal decision x_0 is strictly positive.

A unique maximum:

First order optimum condition:

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T \frac{dT}{dx_0} + c_{x_T} \frac{dx_T}{dx_0} = 0 \tag{105}$$

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T \frac{dT\left(x_0, y_0, a, b\right)}{dx_0} + c_{x_T} \frac{dx_T\left(x_0, y_0, a, b\right)}{dx_0} = 0 \tag{106}$$

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T \frac{dT}{dx_0} + c_{x_T} \frac{dx_T}{dx_0} = 0 \tag{107}$$

$$\frac{d^2\pi}{dx_0{}^2} = -\frac{d^2C}{dx_0{}^2} - c_T \frac{d^2T}{dx_0{}^2} + c_{x_T} \frac{d^2x_T}{dx_0{}^2} \tag{108}$$

$$\left(\frac{d^2C}{dx_0{}^2} \geq 0 \wedge c_T > 0 \wedge \frac{d^2T}{dx_0{}^2} > 0 \wedge c_{x_T} > 0 \wedge \frac{d^2x_T}{dx_0{}^2} < 0 \right) \Rightarrow \frac{d^2\pi}{dx_0{}^2} < 0 \tag{109}$$

Hence, the solution of the first order optimum condition represents a <u>unique maximum</u> of the objective function.

Comparative statics analysis:

The cost per day

Now, we determine how parameter changes affect the optimal deployment decision:

With comparative statics analysis, we see how the optimum is maintained when different possible parameter changes take place. First, the cost per day of the battle is adjusted. The first order optimum condition is differentiated with respect to the optimal value of x_0, denoted x_0^*, and c_T:

$$d\left(\frac{d\pi}{dx_0}\right) = \frac{d^2\pi}{dx_0^{\ 2}} dx_0^{\ *} - \frac{dT}{dx_0} dc_T = 0 \qquad (110)$$

$$\frac{d^2\pi}{dx_0^{\ 2}} dx_0^{\ *} = \frac{dT}{dx_0} dc_T \qquad (111)$$

$$\frac{dx_0^{\ *}}{dc_T} = \frac{\left(\frac{dT}{dx_0}\right)}{\left(\frac{d^2\pi}{dx_0^{\ 2}}\right)} = \frac{(<0)}{(<0)} > 0 \qquad (112)$$

Hence, if the cost per day before the victory increases, then the optimal deployment level increases. This is understandable, since the process will end more rapidly if the initial number of units is larger.

The cost per unit

$$d\left(\frac{d\pi}{dx_0}\right) = \frac{d^2\pi}{dx_0^{\ 2}} dx_0^{\ *} + \frac{dx_T}{dx_0} dc_{x_T} = 0 \quad (113)$$

$$\frac{d^2\pi}{dx_0^{\ 2}} dx_0^{\ *} = -\frac{dx_T}{dx_0} dc_{x_T} \quad (114)$$

$$\frac{dx_0^{\ *}}{dc_{x_T}} = \frac{\left(-\frac{dx_T}{dx_0}\right)}{\left(\frac{d^2\pi}{dx_0^{\ 2}}\right)} = \frac{(<0)}{(<0)} > 0 \quad (115)$$

The result shows that if the cost per unit of killed or wounded troops with equipment increases, then the optimal deployment level increases.

This is understandable, since the number of surviving units is an increasing function of the initial number of units.

Attrition coefficient a

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T \frac{dT(x_0, y_0, a, b)}{dx_0} + c_{x_T} \frac{dx_T(x_0, y_0, a, b)}{dx_0} = 0 \qquad (116)$$

$$\frac{dT}{dx_0} = \frac{-y_0}{bx_0^{\,2} - ay_0^{\,2}} < 0 \qquad (117)$$

$$\frac{dx_T}{dx_0} = \frac{\sqrt{b}\, x_0}{\sqrt{bx_0^{\,2} - ay_0^{\,2}}} > 0 \qquad (118)$$

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T \left(\frac{-y_0}{bx_0^{\,2} - ay_0^{\,2}} \right) + c_{x_T} \left(\frac{\sqrt{b}\, x_0}{\sqrt{bx_0^{\,2} - ay_0^{\,2}}} \right) = 0 \qquad (119)$$

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T\left(\frac{bx_0{}^2 - ay_0{}^2}{-y_0}\right)^{-1} + c_{x_T}\left(\frac{\sqrt{bx_0{}^2 - ay_0{}^2}}{\sqrt{b}\,x_0}\right)^{-1} = 0 \tag{120}$$

$$\frac{d^2\pi}{dx_0 da} = -c_T\,(-1)\left(\frac{bx_0{}^2 - ay_0{}^2}{-y_0}\right)^{-2}\left(-y_0{}^2\right) + c_{x_T}\,(-1)\left(\frac{\sqrt{bx_0{}^2 - ay_0{}^2}}{\sqrt{b}\,x_0}\right)^{-2}\left(\frac{1}{2}\right)\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\left(-y_0{}^2\right) \tag{121}$$

$$\frac{d^2\pi}{dx_0 da} = y_0{}^2\left(c_T\left(\frac{bx_0{}^2 - ay_0{}^2}{y_0}\right)^{-2} + c_{x_T}\left(\frac{\sqrt{bx_0{}^2 - ay_0{}^2}}{\sqrt{b}\,x_0}\right)^{-2}\left(\frac{1}{2}\right)\left(bx_0{}^2 - ay_0{}^2\right)^{-\frac{1}{2}}\right) > 0 \tag{122}$$

$$d\left(\frac{d\pi}{dx_0}\right) = \frac{d^2\pi}{dx_0{}^2}dx_0{}^* + \frac{d^2\pi}{dx_0 da}da = 0 \tag{123}$$

$$\frac{d^2\pi}{dx_0^{\ 2}} dx_0^{\ *} = -\frac{d^2\pi}{dx_0 da} da \qquad (124)$$

$$\frac{dx_0^{\ *}}{da} = \frac{\left(-\frac{d^2\pi}{dx_0 da}\right)}{\left(\frac{d^2\pi}{dx_0^{\ 2}}\right)} = \frac{(<0)}{(<0)} > 0 \qquad (125)$$

Hence, if the attrition coefficient a increases,

then the optimal deployment increases.

Attrition coefficient b

$$\frac{d\pi}{dx_0} = -\frac{dC}{dx_0} - c_T\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{-y_0}\right)^{-1} + c_{x_T}\left(\frac{\sqrt{bx_0^{\ 2} - ay_0^{\ 2}}}{\sqrt{b}\,x_0}\right)^{-1} = 0 \qquad (126)$$

$$\frac{d^2\pi}{dx_0 db} = -c_T(-1)\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{-y_0}\right)^{-2}\left(x_0^{\ 2}\right) + c_{x_T}(-1)\left(\frac{\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{b}\right)^{\frac{1}{2}}}{x_0}\right)^{-2}\left(\frac{1}{2}\right)\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{b}\right)^{\frac{3}{2}}\left(x_0^{\ 2}\right) \qquad (127)$$

$$\frac{d^2\pi}{dx_0 db} = -c_T\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{y_0}\right)^{-2}\left(x_0^{\ 2}\right) - c_{x_T}\left(\frac{\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{b}\right)^{\frac{1}{2}}}{x_0}\right)^{-2}\left(\frac{1}{2}\right)\left(\frac{bx_0^{\ 2} - ay_0^{\ 2}}{b}\right)^{\frac{3}{2}}\left(x_0^{\ 2}\right) < 0 \qquad (128)$$

$$\frac{d^2\pi}{dx_0 db} = x_0{}^2 \left(\left(-c_T\right) \frac{y_0{}^2}{\left(bx_0{}^2 - ay_0{}^2\right)^2} - \left(c_{x_T}\right) \frac{x_0{}^2}{\left(\frac{bx_0{}^2 - ay_0{}^2}{b}\right)} \left(\frac{1}{2}\right) \left(\frac{bx_0{}^2 - ay_0{}^2}{b}\right)^{\frac{3}{2}} \right) \tag{129}$$

$$\frac{d^2\pi}{dx_0 db} = -x_0{}^2 \left(\frac{c_T y_0{}^2}{\left(bx_0{}^2 - ay_0{}^2\right)^2} + \frac{c_{x_T} x_0{}^2}{2} \sqrt{\frac{bx_0{}^2 - ay_0{}^2}{b}} \right) < 0 \tag{130}$$

$$d\left(\frac{d\pi}{dx_0}\right)=\frac{d^2\pi}{dx_0{}^2}dx_0{}^*+\frac{d^2\pi}{dx_0 db}db=0 \tag{131}$$

$$\frac{d^2\pi}{dx_0{}^2}dx_0{}^*=-\frac{d^2\pi}{dx_0 db}db \tag{132}$$

$$\frac{dx_0{}^*}{db}=\frac{\left(-\frac{d^2\pi}{dx_0 db}\right)}{\left(\frac{d^2\pi}{dx_0{}^2}\right)}=\frac{(>0)}{(<0)}<0 \tag{133}$$

Hence, if the attrition coefficient b increases, then the optimal deployment decreases.

This is also illustrated in Figure 21.

Numerical optimization with known attrition parameters

Numerical Model 1:

Continuous optimization model with Newton Raphson iteration. CASE 0.

F	cx0	G	cT	cK
1000	1.000	200000	730.000	2.000

a	b	y0	x0_0
0.053470	0.010450	18000	90000

Parameters and initial guess

Newton Raphson iteration

n	x0	T	K	dx0
0	90000			
1	75787	25.395	11866	-14213.471601
2	71793	27.214	12662	-3993.160551
3	69728	28.277	13123	-2065.419803
4	65658	30.680	14149	-4069.855630
5	66147	30.366	14016	488.771454
6	66156	30.360	14014	9.613701
7	66156	30.360	14014	0.003585
8	66156	30.360	14014	0.000000

Initial guess

Optimal solution

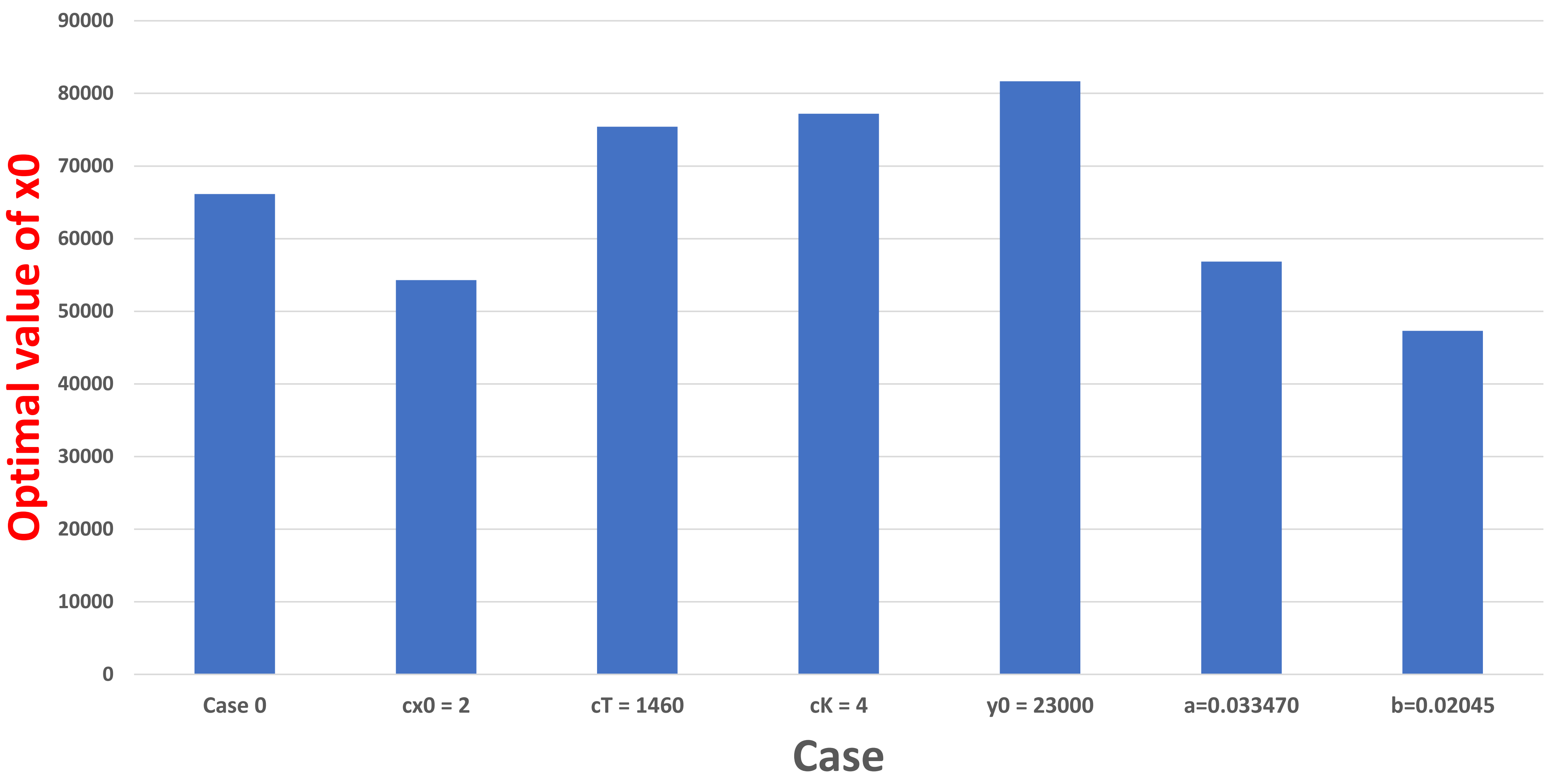


Figure 22.

The optimal values of x_0, according to Numerical model 1, in alternative cases.

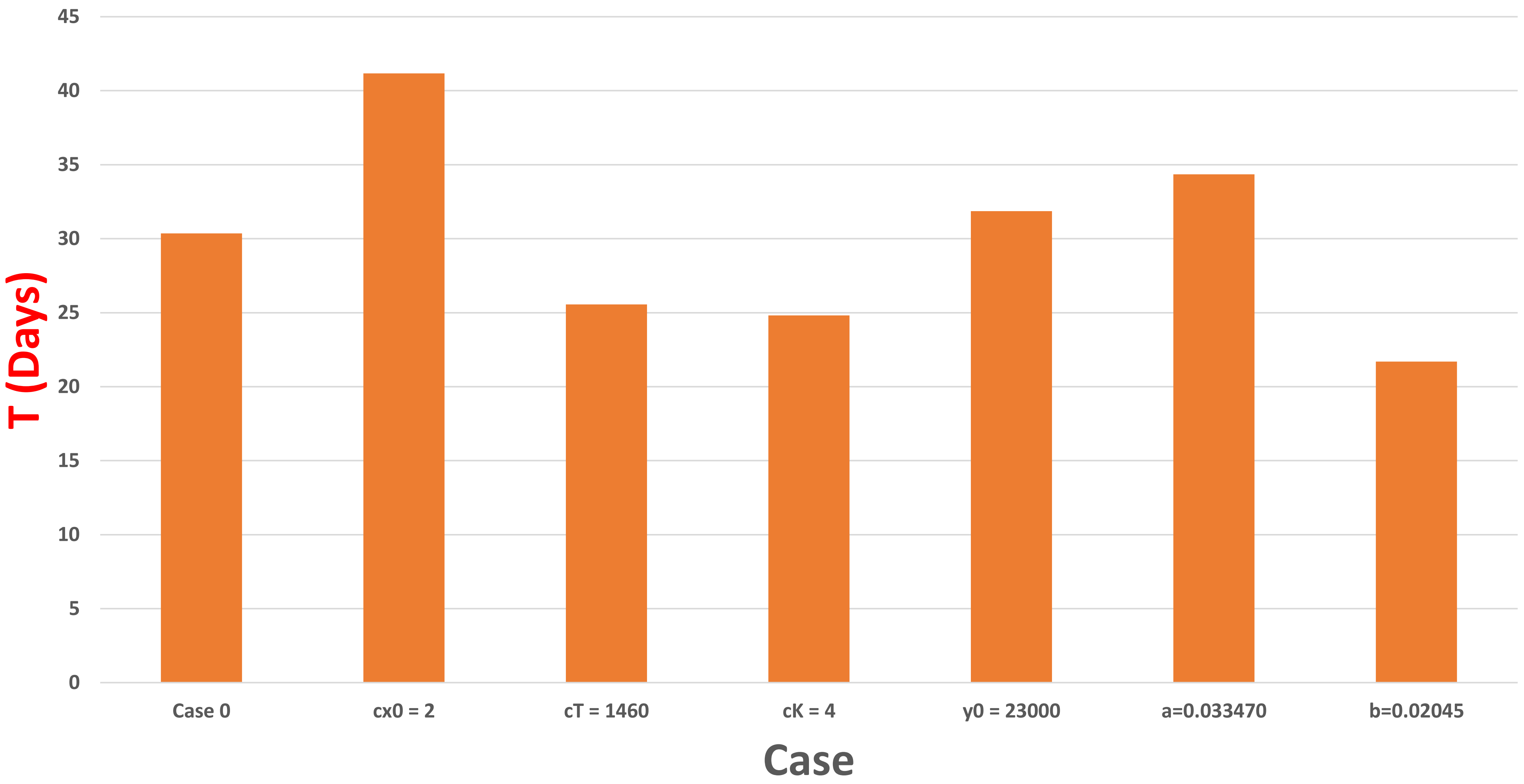


Figure 23.

The optimal values of T, the day of the victory, according to Numerical model 1, in alternative cases.

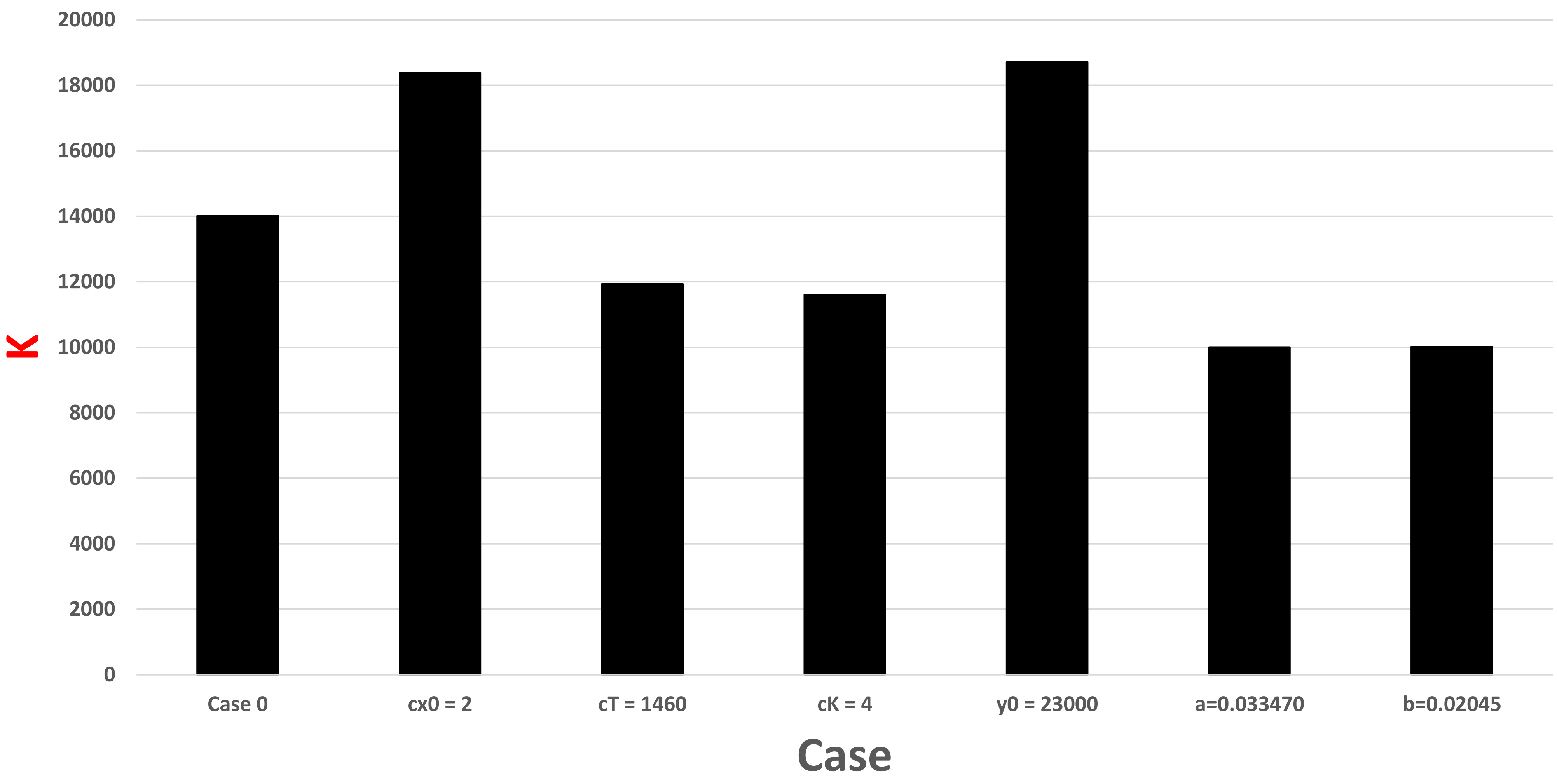


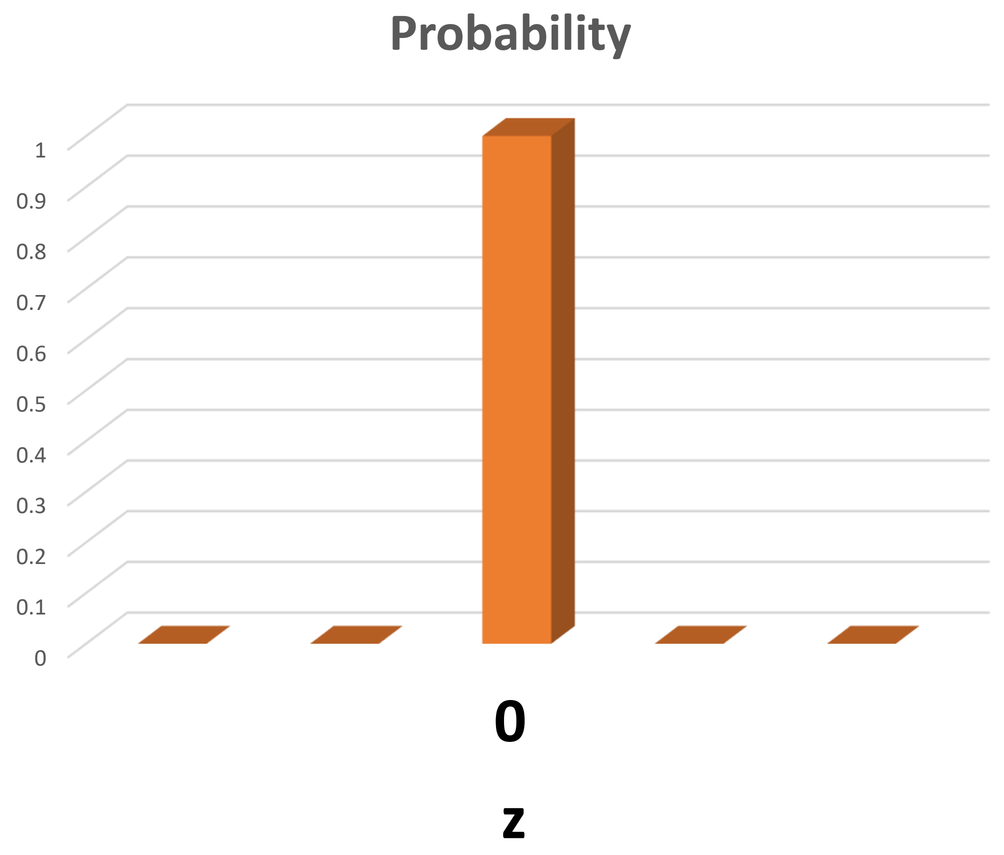
Figure 24.

The optimal values of K, the number of killed and wounded soldiers, according to Numerical model 1, in alternative cases.

Numerical optimization with stochastic attrition parameters

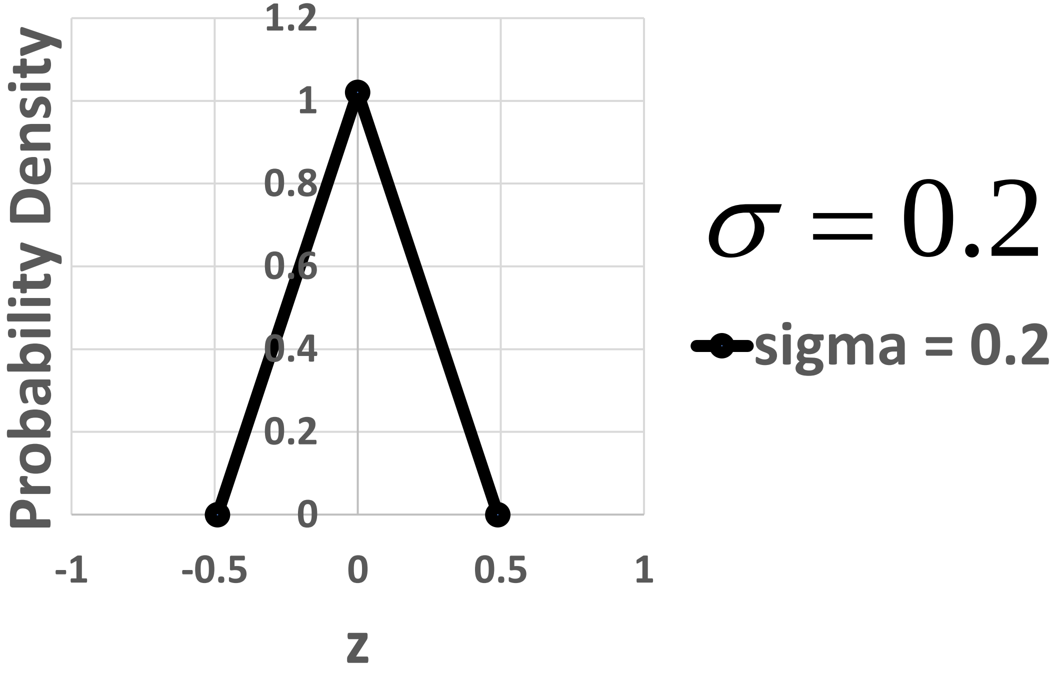
z *= the stochastic deviation from the expected value (of an attrition parameter).*

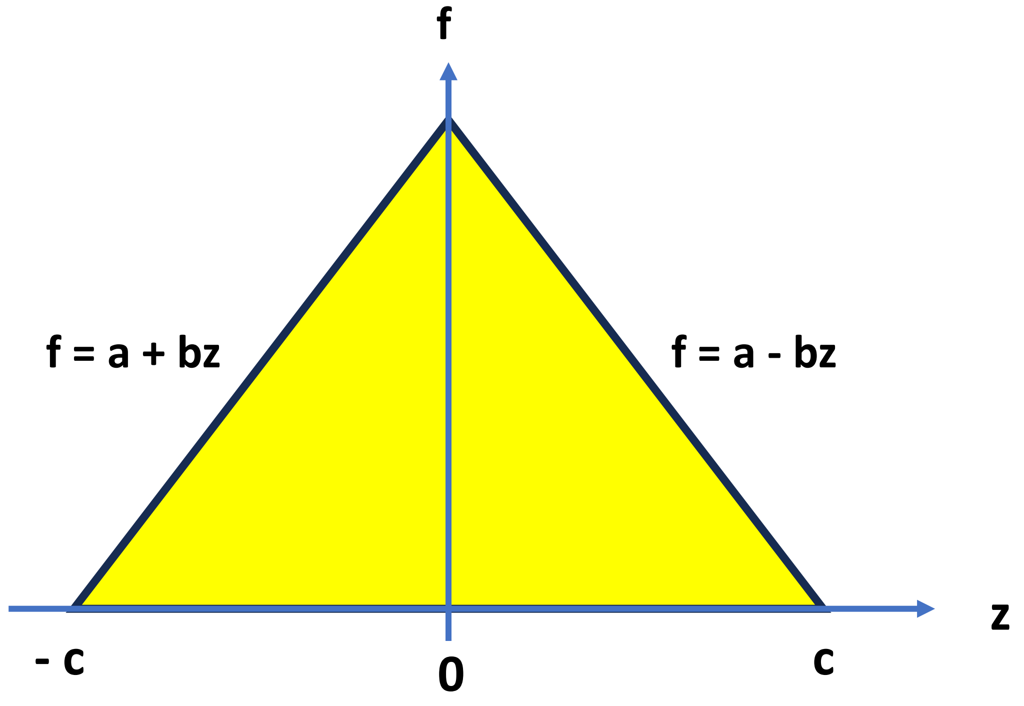
When the operation is planned, the true value of **z** *is not known, but the probability distribution "and/or" the probability density function, "are/is" known.*



Assumption in deterministic analysis.

Assumption in stochastic analysis (with continuous z).





In the first quadrant, the function f reaches the z-axes where z = c.

a – bc = 0

bc = a

b = a/c

f = a + (a/c)z

f = a(1 + 1/c)z

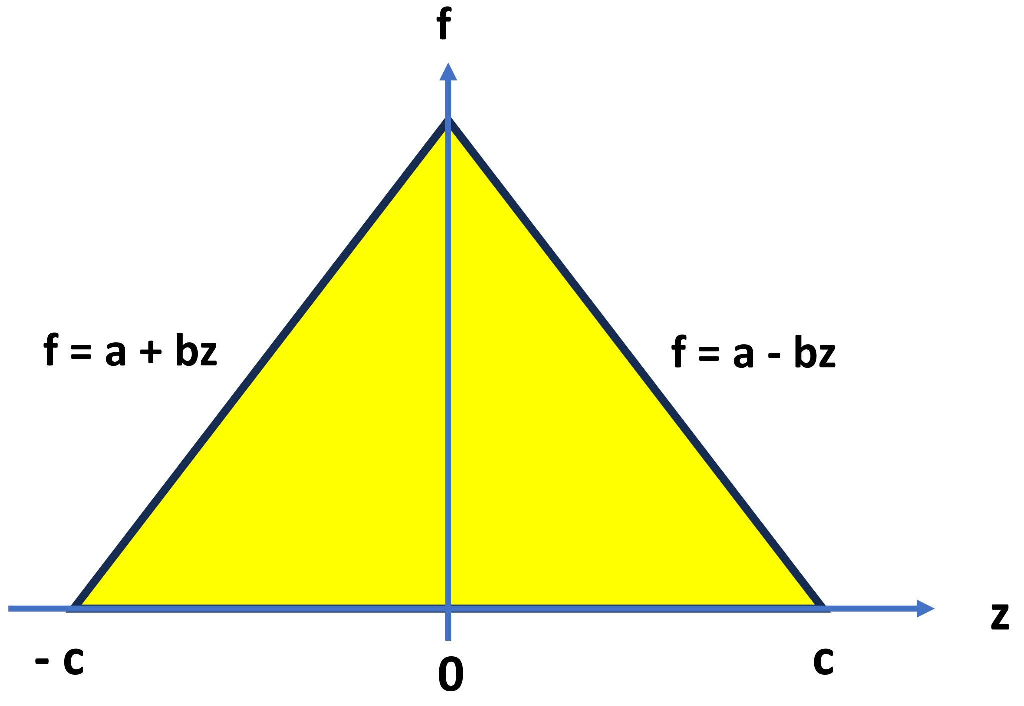
The total probability (yellow) is 1.

ac = 1

a = 1/c

$$f = \frac{1}{c} + \frac{1}{c^2} z$$

(In the first quadrant)



$$f = \frac{1}{c} + \frac{1}{c^2} z$$

(In the first quadrant)

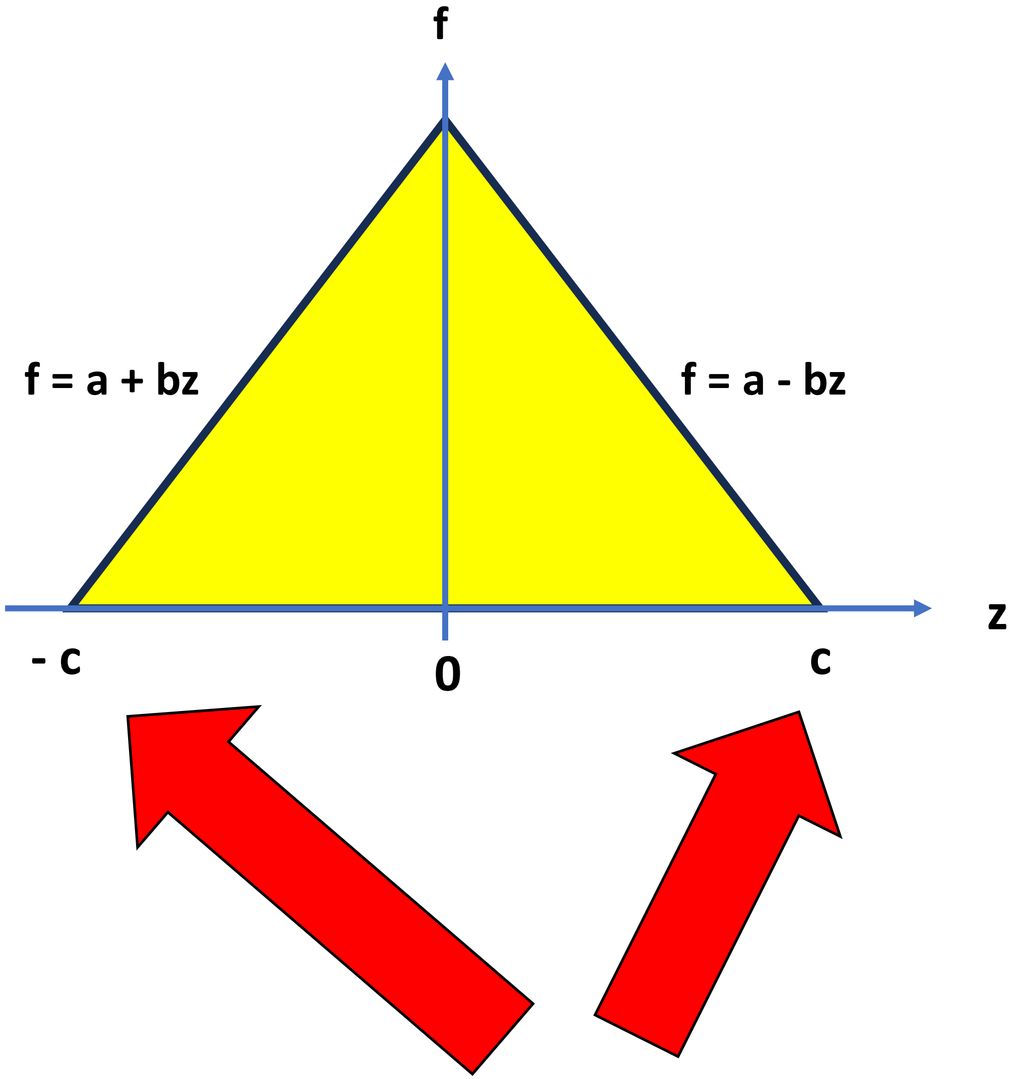
$$f = \frac{1}{c} + \frac{1}{c^2} z$$

$$\sigma^2 = 2\int_0^c z^2 \left(\frac{1}{c} - \frac{1}{c^2} z \right) dz$$

$$\sigma^2 = \frac{2}{c}\int_0^c z^2 \left(1 - \frac{1}{c} z \right) dz$$

$$\sigma^2 = \frac{2}{c}\int_0^c z^2 \, dz - \frac{2}{c^2}\int_0^c z^3 \, dz$$

$$\sigma^2 = \frac{2}{c}\left[\frac{z^3}{3} \right]_0^c - \frac{2}{c^2}\left[\frac{z^4}{4} \right]_0^c$$



$$\left(\sigma = 0.2\right) \Rightarrow \left(c \approx 0.489898\right)$$

$$\sigma^2 = \frac{2}{c}\left[\frac{z^3}{3}\right]_0^c - \frac{2}{c^2}\left[\frac{z^4}{4}\right]_0^c$$

$$\sigma^2 = \frac{2}{3c}c^3 - \frac{1}{2c^2}c^4$$

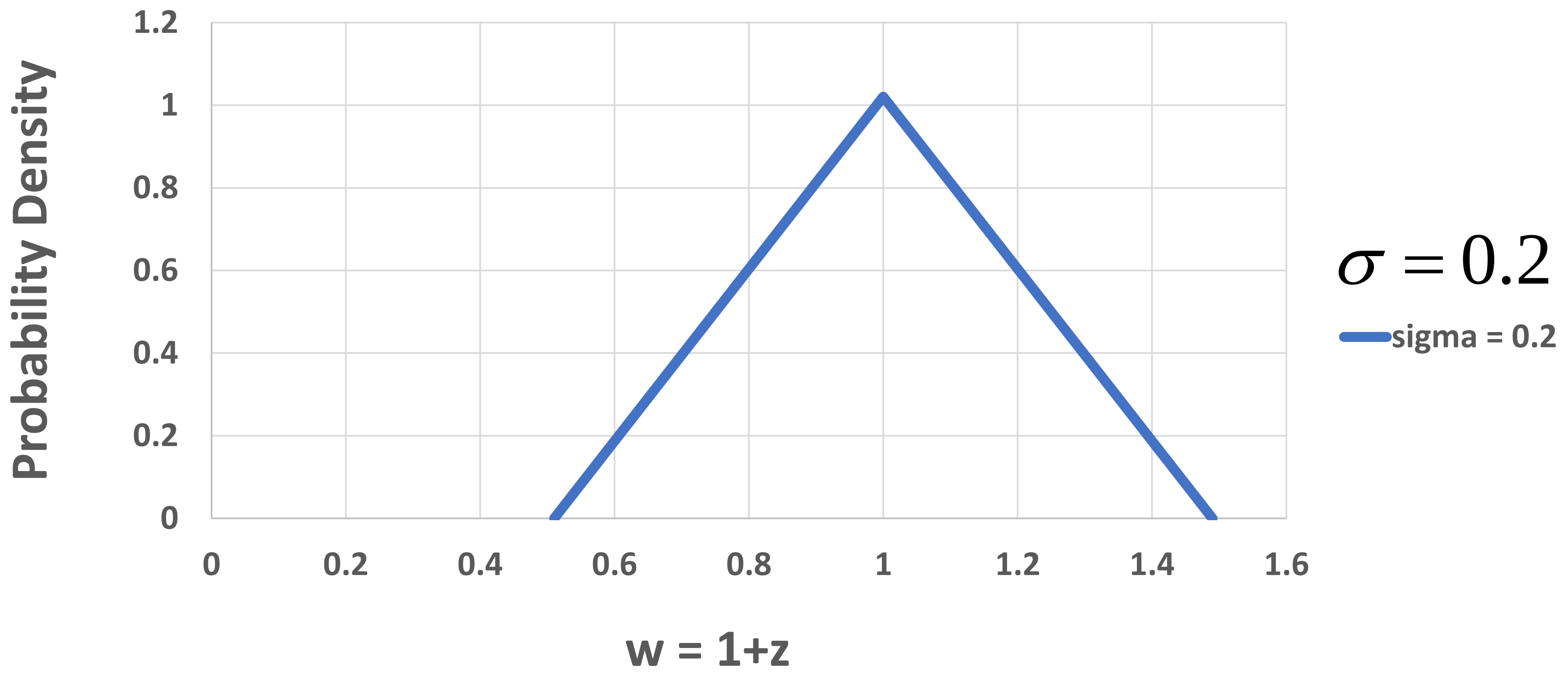
$$\sigma^2 = \frac{2}{3}c^2 - \frac{1}{2}c^2$$

$$\sigma^2 = \frac{4}{6}c^2 - \frac{3}{6}c^2$$

$$\sigma^2 = \frac{1}{6}c^2$$

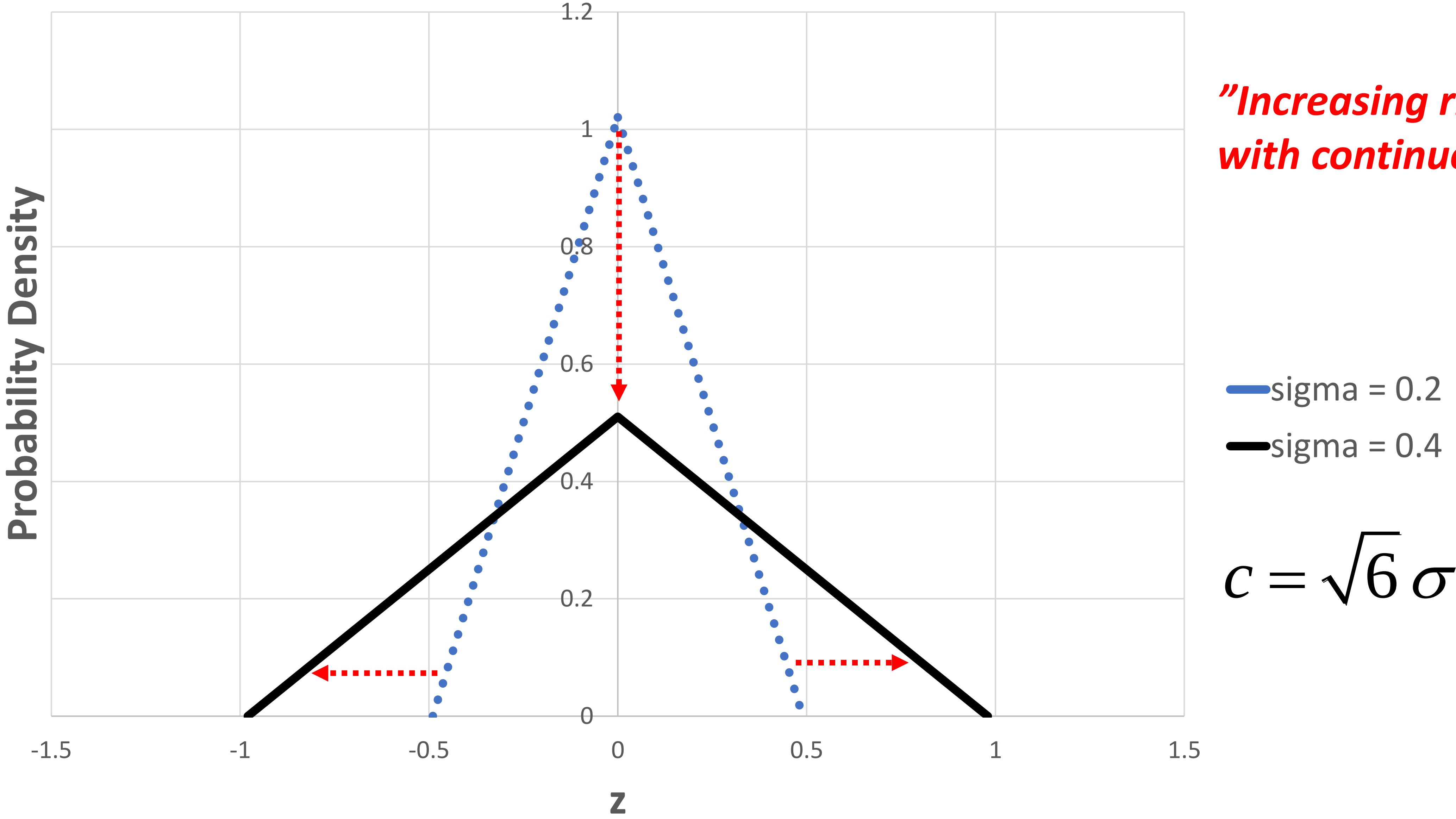
$$c^2 = 6\sigma^2$$

$$c = \sqrt{6}\,\sigma$$



After the deployment, when the operation starts, z is observed.

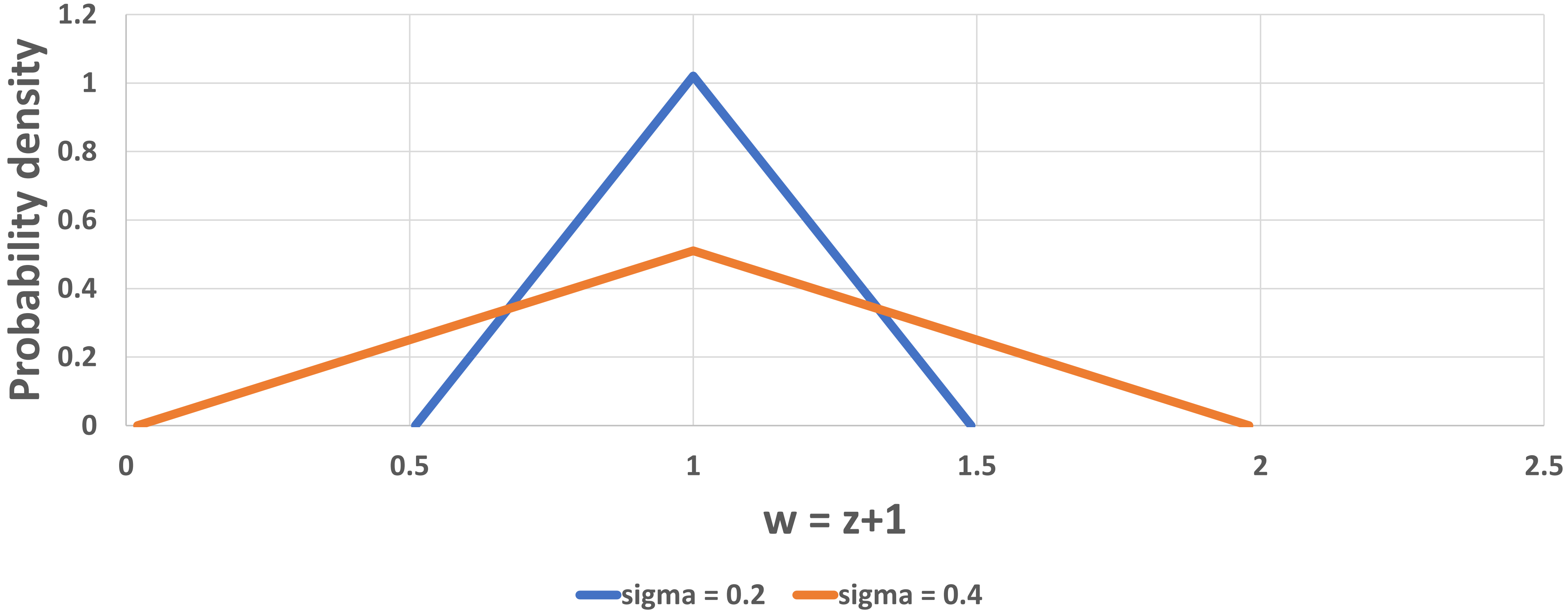
Then, the expected value of the attrition parameter is multiplied by w = (1+z), to get the true value of the attrition parameter.



"Increasing risk" with continuous z

$$c = \sqrt{6}\,\sigma$$

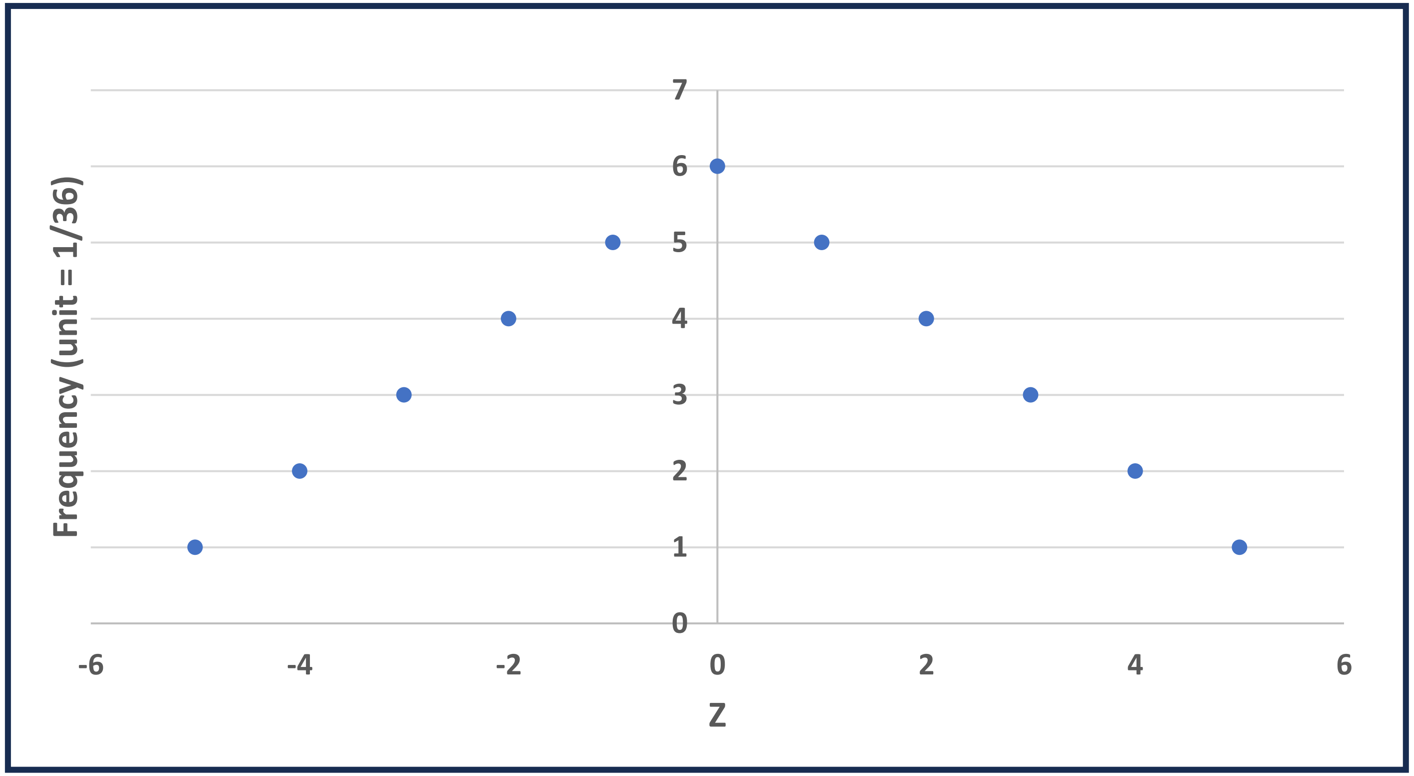
Examples of the attrition parameter multiplier probability density function.



Assumptions in stochastic analysis (with discrete z).

First, we consider this discrete approximation of a triangular probability distribution:

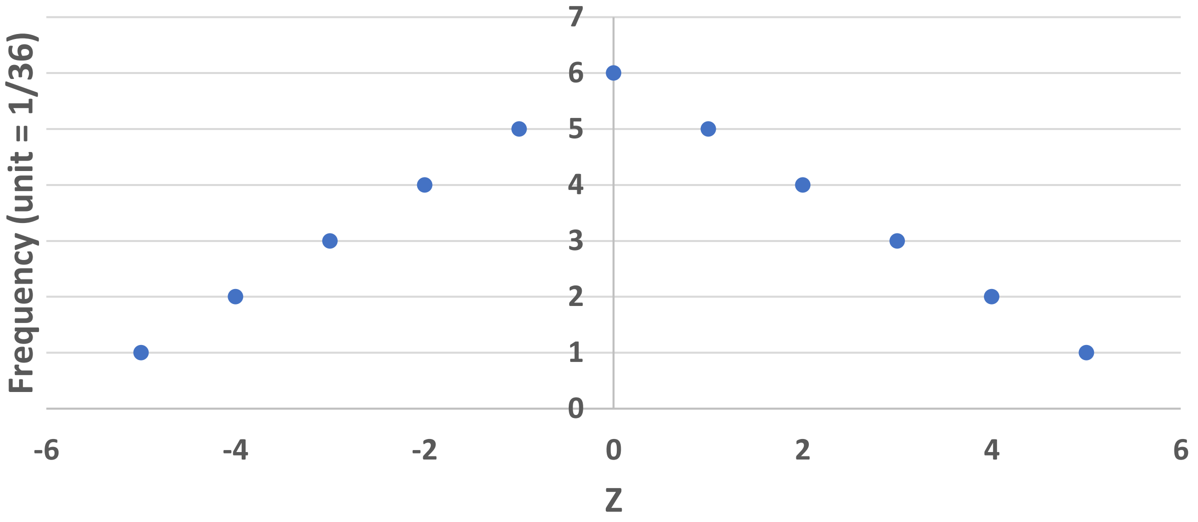
z	freq	Probability
-5	1	0.027778
-4	2	0.055556
-3	3	0.083333
-2	4	0.111111
-1	5	0.138889
0	6	0.166667
1	5	0.138889
2	4	0.111111
3	3	0.083333
4	2	0.055556
5	1	0.027778

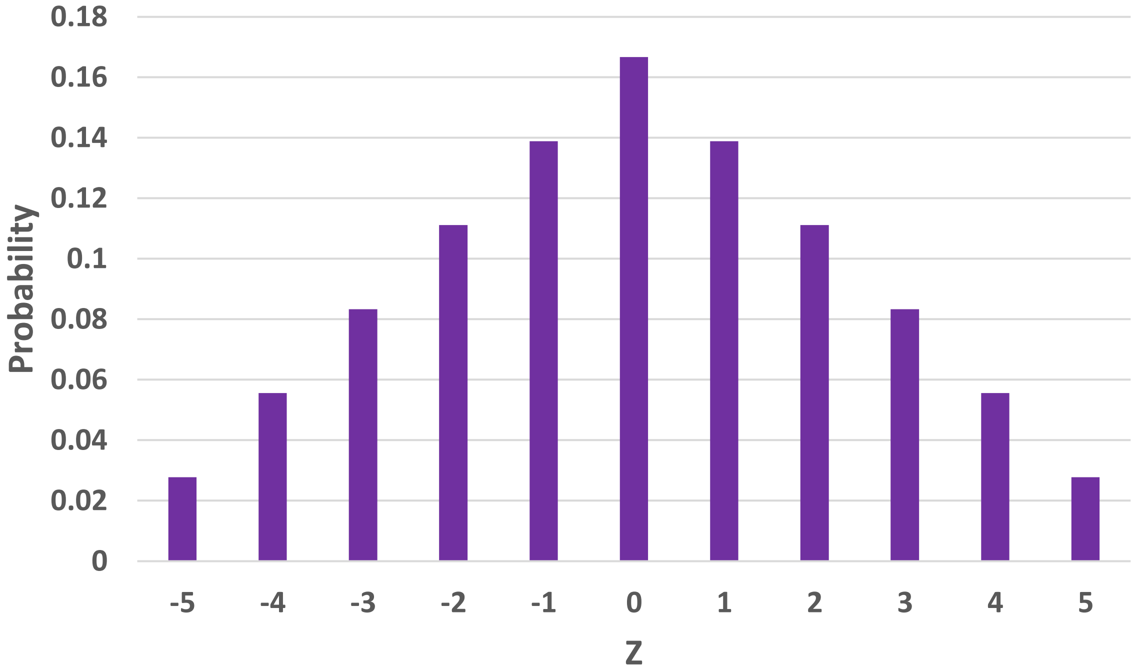


The total probability = 1.

P(z) = (1/36) x (6-z) for 0<=z<=5
P(z) = (1/36) x (6+z) for 5<=z<0

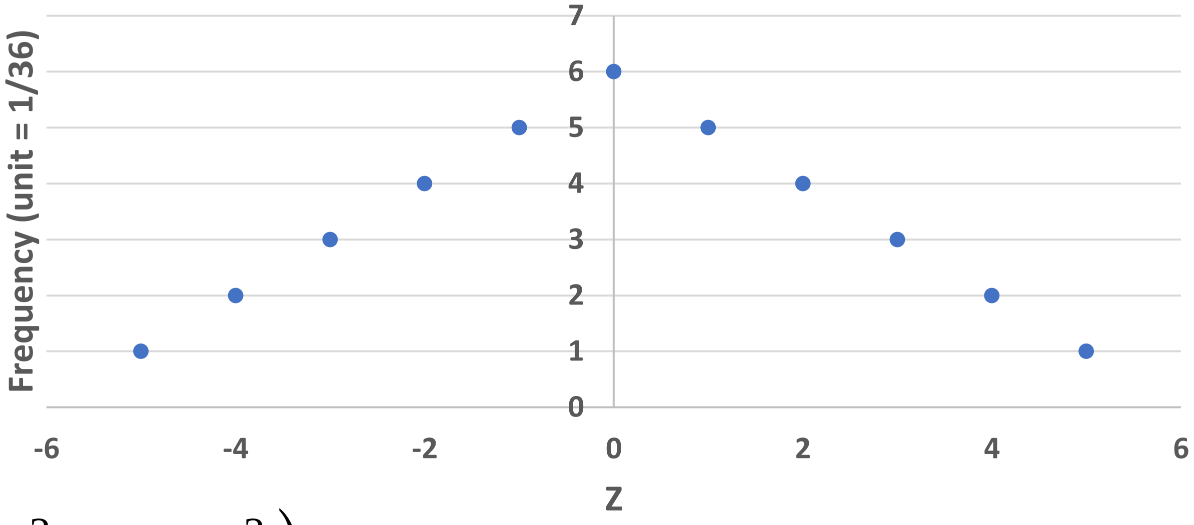
z	Freq	Probability
-5	1	0.027778
-4	2	0.055556
-3	3	0.083333
-2	4	0.111111
-1	5	0.138889
0	6	0.166667
1	5	0.138889
2	4	0.111111
3	3	0.083333
4	2	0.055556
5	1	0.027778





Now, we multiply each z-value by k, and investigate the variance σ^2:

$$\sigma^2 = \frac{2}{36}\left(1\times(5k)^2 + 2\times(4k)^2 + 3\times(3k)^2 + 4\times(2k)^2 + 5\times(1k)^2\right)$$



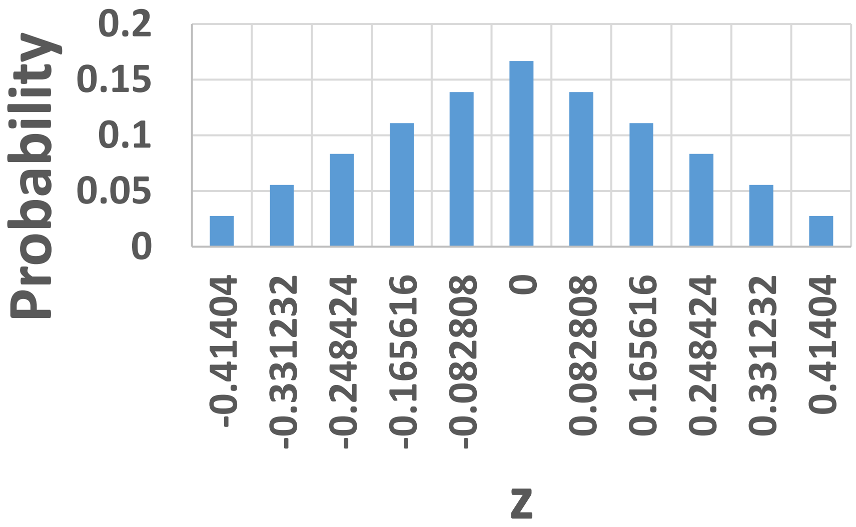
$$\sigma^2 = \frac{k^2}{18}\left(5^2 + 2\times 4^2 + 3\times 3^2 + 4\times 2^2 + 5\times 1^2\right)$$

$$\sigma^2 = \frac{105}{18}k^2 \Rightarrow k^2 = \frac{18}{105}\sigma^2 \Rightarrow k = \sqrt{\frac{18}{105}}\sigma$$

If we know the variance, we know k.

$$k = \sqrt{\frac{18}{105}}\,\sigma \approx 0.082808$$

Discrete Z



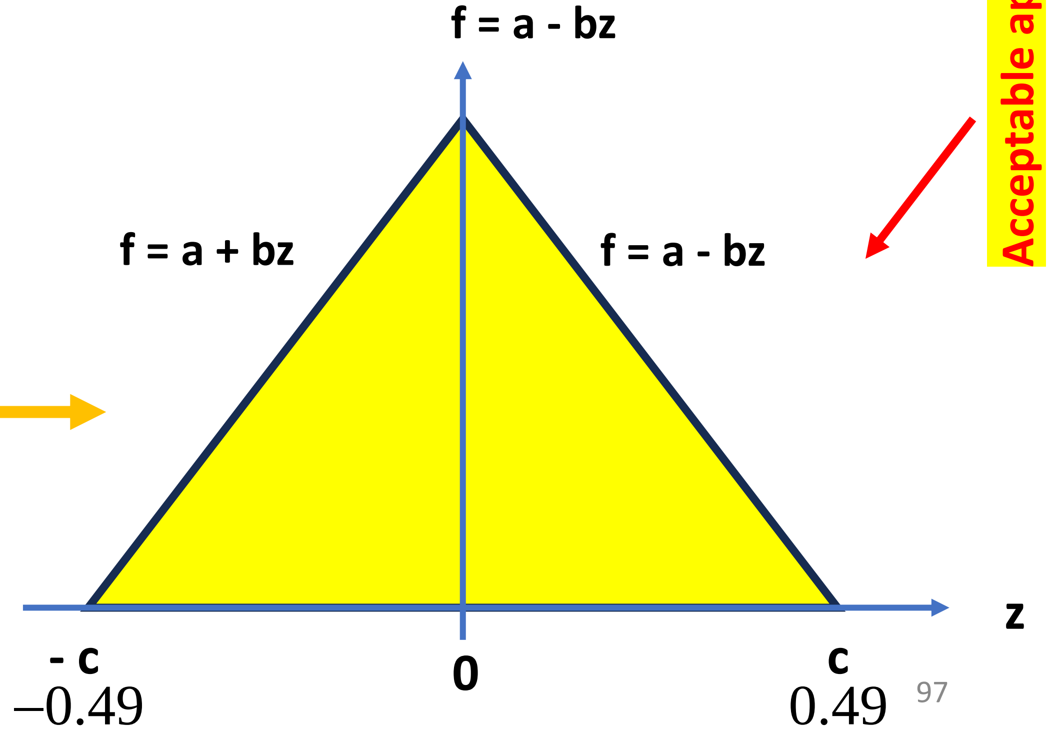
$$\sigma = 0.2$$

z	Probability
-0.41404	0.027778
-0.33123	0.055556
-0.24842	0.083333
-0.16562	0.111111
-0.08281	0.138889
0	0.166667
0.082808	0.138889
0.165616	0.111111
0.248424	0.083333
0.331232	0.055556
0.41404	0.027778

$$c = \sqrt{6}\,\sigma$$

$$c \approx 0.490$$

Continuous Z



Acceptable approximation.

First, we recall the deterministic version of the problem:

$$\max_{x_0} \pi = -C(x_0) + G - c_T T(x_0, y_0, a, b) - c_{x_T}(x_0 - x_T(x_0, y_0, a, b))$$

T **is affected by a and b.**

x_T **is affected by a and b.**

$$\max_{x_0} \pi = -C(x_0) + G - c_T \frac{LN\left(\dfrac{x_0 + \sqrt{\dfrac{a}{b}}\, y_0}{x_0 - \sqrt{\dfrac{a}{b}}\, y_0}\right)}{2r} - c_{x_T}\left(x_0 - \sqrt{\frac{b x_0^{\;2} - a y_0^{\;2}}{b}}\right)$$

The deterministic version was:

$$\max_{x_0} \pi = -C(x_0) + G - c_T T(x_0, y_0, a, b) - c_{x_T}\left(x_0 - x_T(x_0, y_0, a, b)\right)$$

Now, however, we want to maximize the expected total result.

The parameters a and b are not known before x_0 has been decided.

Stochastic problem with discrete a and b:

$$\max_{x_0} E(R_x(x_0)) = -C(x_0) + \sum_a \sum_b R(x_0; a, b) P(a, b)$$ ***General***

$$\max_{x_0} E(R_x(x_0)) = -C(x_0) + \sum_a \sum_b R(x_0; a, b) P_a(a) P_b(b)$$ ***Corr = 0***

The deterministic version was:

$$\max_{x_0} \pi = -C(x_0) + G - c_T T(x_0, y_0, a, b) - c_{x_T}\left(x_0 - x_T(x_0, y_0, a, b)\right)$$

Now, however, we want to maximize the expected total result.

The parameters a and b are not known before x_0 has been decided.

Stochastic problem with continuous a and b:

$$\max_{x_0} E(R_x(x_0)) = -C(x_0) + \int\int R(x_0; a, b) f(a, b)\, db\, da$$ ***General***

$$\max_{x_0} E(R_x(x_0)) = -C(x_0) + \int\int R(x_0; a, b) f_b(b) f_a(a)\, db\, da$$ ***Corr = 0***

<u>Numerical Model 2:</u>

Discrete optimization model with stochastic attrition coefficients:

This model, partly based on the analytical derivations presented in the earlier sections, determines the optimal decisions and consequences, via numerical calculations, for alternative deployment levels.

The optimal value of the objective functions is defined as the highest value of the investigated alternatives.

RESULTS FROM STBLPL 230919_2044 by Peter Lohmander

PARAMETERS =

R_Wx1 = 300000 ← **G**

R_Wx2 = 0

R_tF = 0

R_x0 = -2 ← **-1 x (Marginal Deployment Cost)**

R_KIAx = 0

a_mean = .0544

b_mean = .0106

a_sigma = .2

b_sigma = .2

x0	y0	E_xF	E_yF	E_KIAx	E_KIAy	E_Wx	E_Wy	E_tF	E_Rx	E_Ry
0	21500	0	21500	0	0	0.0000	1.0000	1	0	-6
5000	21500	0	21354	5000	146	0.0000	1.0000	5	-10	-7
10000	21500	0	20967	10000	533	0.0000	1.0000	10	-20	-7
15000	21500	0	20326	15000	1174	0.0000	1.0000	14	-30	-8
20000	21500	0	19402	20000	2098	0.0000	1.0000	20	-40	-9
25000	21500	0	18148	25000	3352	0.0000	1.0000	26	-50	-10
30000	21500	0	16469	30000	5031	0.0000	1.0000	33	-60	-11
35000	21500	123	14130	34877	7370	0.0123	0.9877	42	-66	-15
40000	21500	1260	10888	38740	10612	0.1011	0.8989	54	-50	-27
45000	21500	5027	7082	39973	14418	0.3218	0.6782	65	7	-59
50000	21500	12269	3724	37731	17776	0.5579	0.4421	67	67	-106
55000	21500	21773	1601	33227	19899	0.8164	0.1836	61	135	-161
60000	21500	31313	580	28687	20920	0.9244	0.0756	53	157	-203
65000	21500	40071	174	24929	21326	0.9807	0.0193	46	164	-236
70000	21500	47937	36	22063	21464	0.9961	0.0039	40	159	-260
75000	21500	55101	3	19899	21497	1.0000	0.0000	35	150	-279
80000	21500	61776	0	18224	21500	1.0000	0.0000	32	140	-295
85000	21500	68129	0	16871	21500	1.0000	0.0000	29	130	-308
90000	21500	74259	0	15741	21500	1.0000	0.0000	27	120	-320
95000	21500	80225	0	14775	21500	1.0000	0.0000	25	110	-331
100000	21500	86064	0	13936	21500	1.0000	0.0000	24	100	-340
105000	21500	91800	0	13200	21500	1.0000	0.0000	23	90	-348
110000	21500	97454	0	12546	21500	1.0000	0.0000	21	80	-356
115000	21500	103040	0	11960	21500	1.0000	0.0000	20	70	-363
120000	21500	108567	0	11433	21500	1.0000	0.0000	19	60	-368
125000	21500	114044	0	10956	21500	1.0000	0.0000	18	50	-375
130000	21500	119481	0	10519	21500	1.0000	0.0000	18	40	-380
135000	21500	124878	0	10122	21500	1.0000	0.0000	17	30	-385
140000	21500	130247	0	9753	21500	1.0000	0.0000	16	20	-389
145000	21500	135583	0	9417	21500	1.0000	0.0000	16	10	-394
150000	21500	140899	0	9101	21500	1.0000	0.0000	15	0	-398

Optimal value of x0 = Opt_x0 = 65000

Optimal value of E_Rx = Opt_E_Rx = 164.212962962963

Optimum

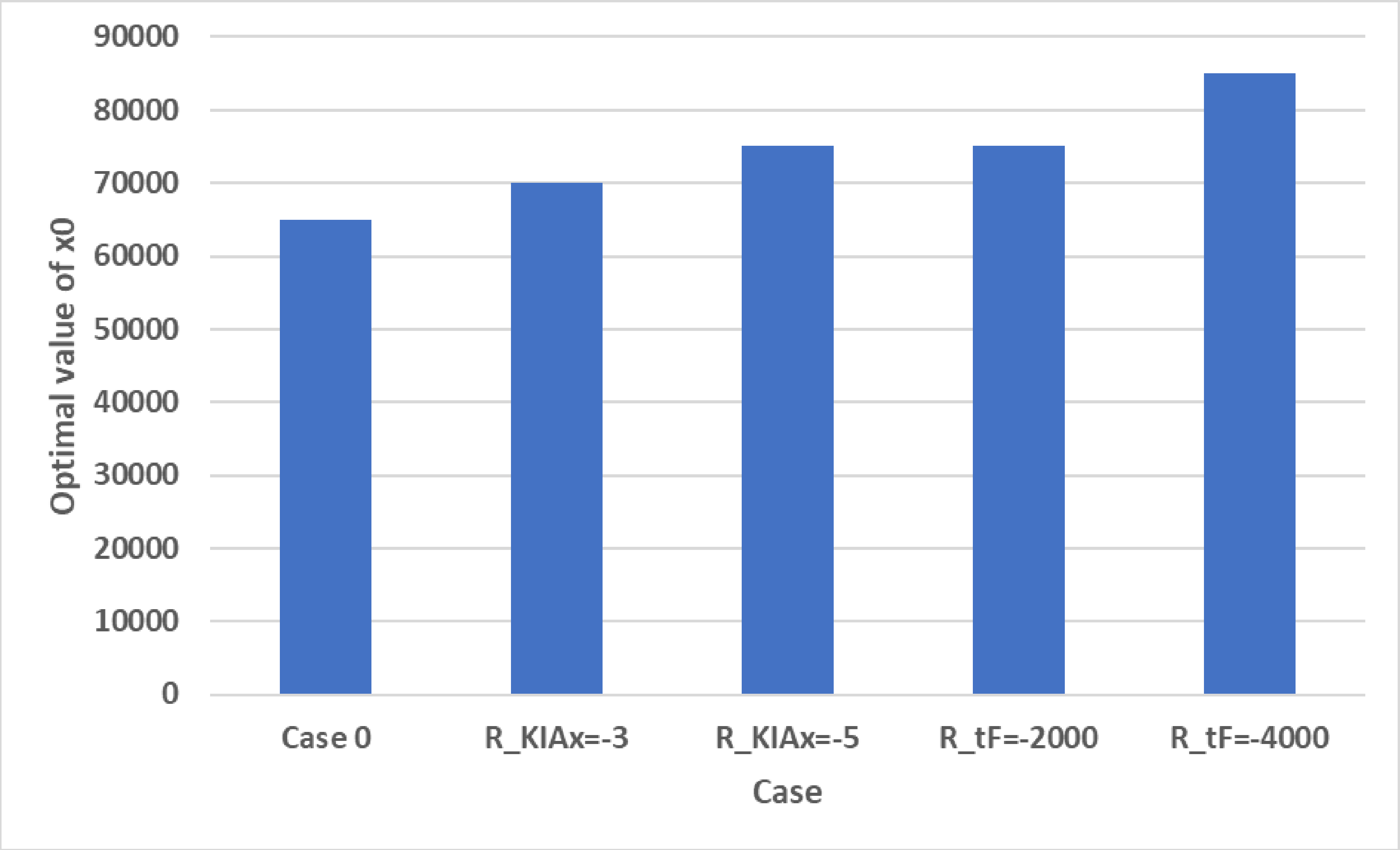
Attrition parameter a:

Expected value 0.0544 and relative standard deviation 0.2

Attrition parameter b:

Expected value 0.0106 and relative standard deviation 0.2

Correlation between a and b is zero.



<u>Figure 25.</u>

The optimal values of x_0, according to Numerical model 2, in alternative cases.

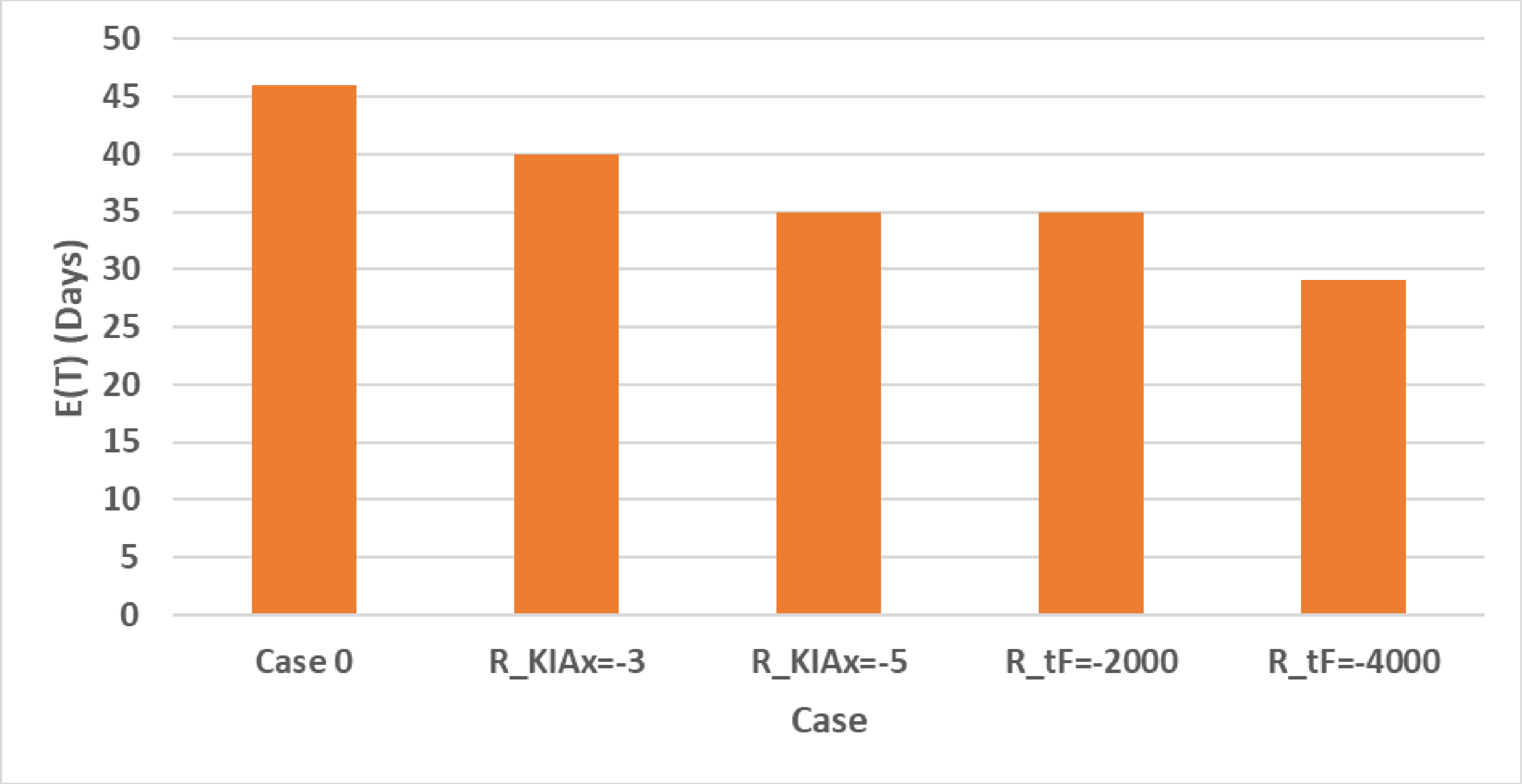


Figure 26.

The optimal expected values of E(T), the day of the victory, according to Numerical model 2, in alternative cases.

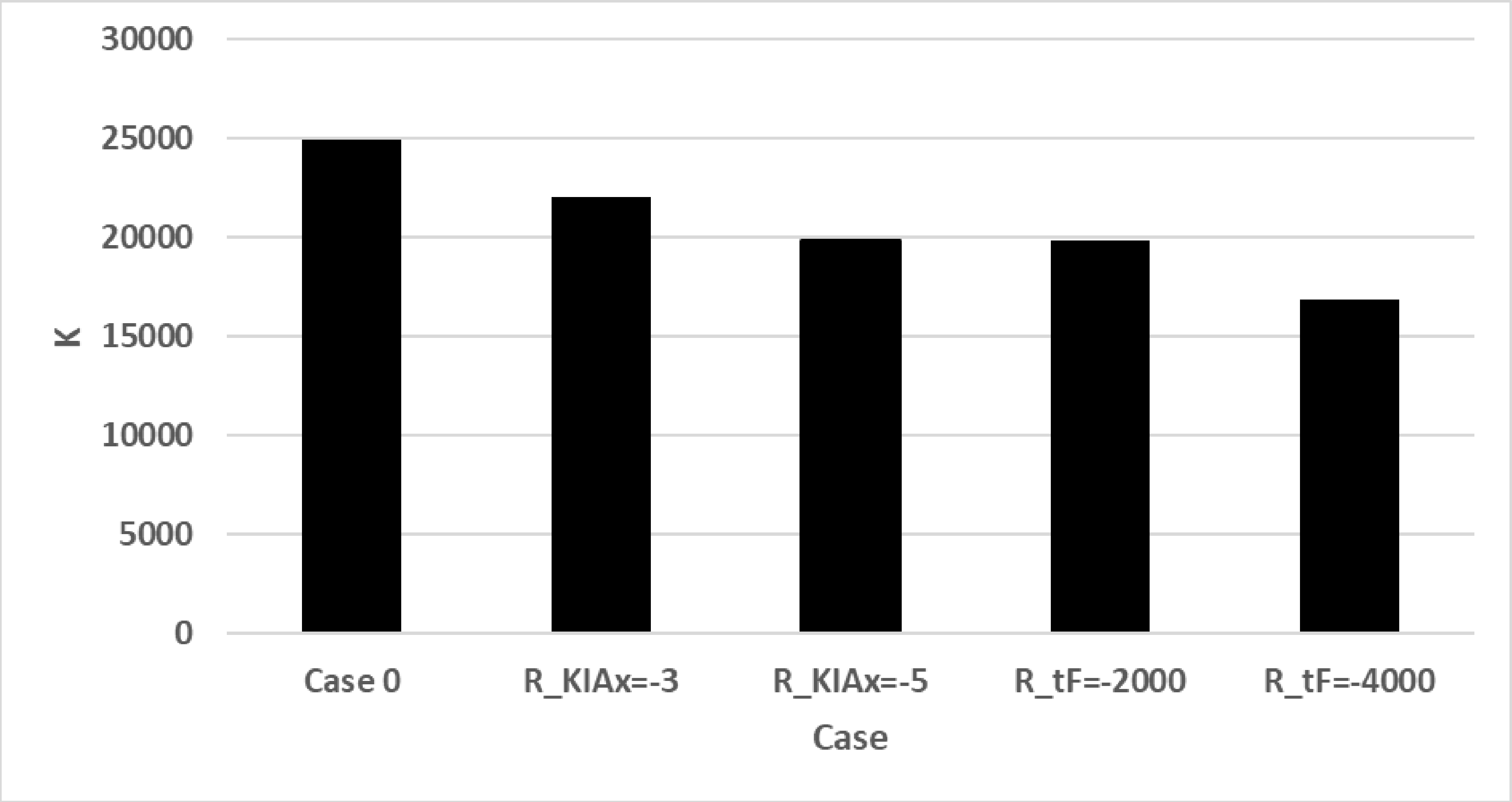


Figure 27.

The optimal expected values of K, (= KIAx), the number of killed and wounded soldiers, according to Numerical model 2, in alternative cases.

Short summary of the results:

The analytical, and the two numerical, methods, all show that the optimal deployment level is

a decreasing function of the marginal deployment cost,

an increasing function of the marginal cost of the time to win the battle,

an increasing function of the marginal cost of killed and wounded soldiers and lost equipment,

an increasing function of the initial size of the opposing army,

an increasing function of the efficiency of the soldiers in the opposing army and

a decreasing function of the efficiency of the soldiers in the deployed army.

With stochastic attrition parameters, the stochastic model also shows that the

probability to win the battle is an increasing function of the size of the deployed army.

When the optimal deployment level is selected, the probability of a victory is usually less than 100%, since it would be too expensive to guarantee a victory with 100% probability.

Optimal Deployment

by
Peter Lohmander

Thank you for your time! Questions?

ICSTC-2024

Tenth International Conference on Statistics for the Twenty-First Century

December 13 - 16, 2024, Special Invited Talk.

Organized by International Statistics Fraternity (ISF) in collaboration with the Department of Statistics, University of Kerala and School of Physical and Mathematical Sciences at the University of Kerala and the American Statistical Association.

Version 241211_1448

Peter Lohmander
Prof Dr

Peter Lohmander Optimal Solutions, Sweden