L4 (By Peter Lohmander 2009-09-23)

Quadratic programming and other multi dimensional nonlinear programming. Focus on problems with concave objective functions and convex feasible sets. General theory, application examples, analytical solutions and numerical solutions via computer programming.
(Portfolio theory according to Markowitz with applications in separate document.)

Background to a particular case

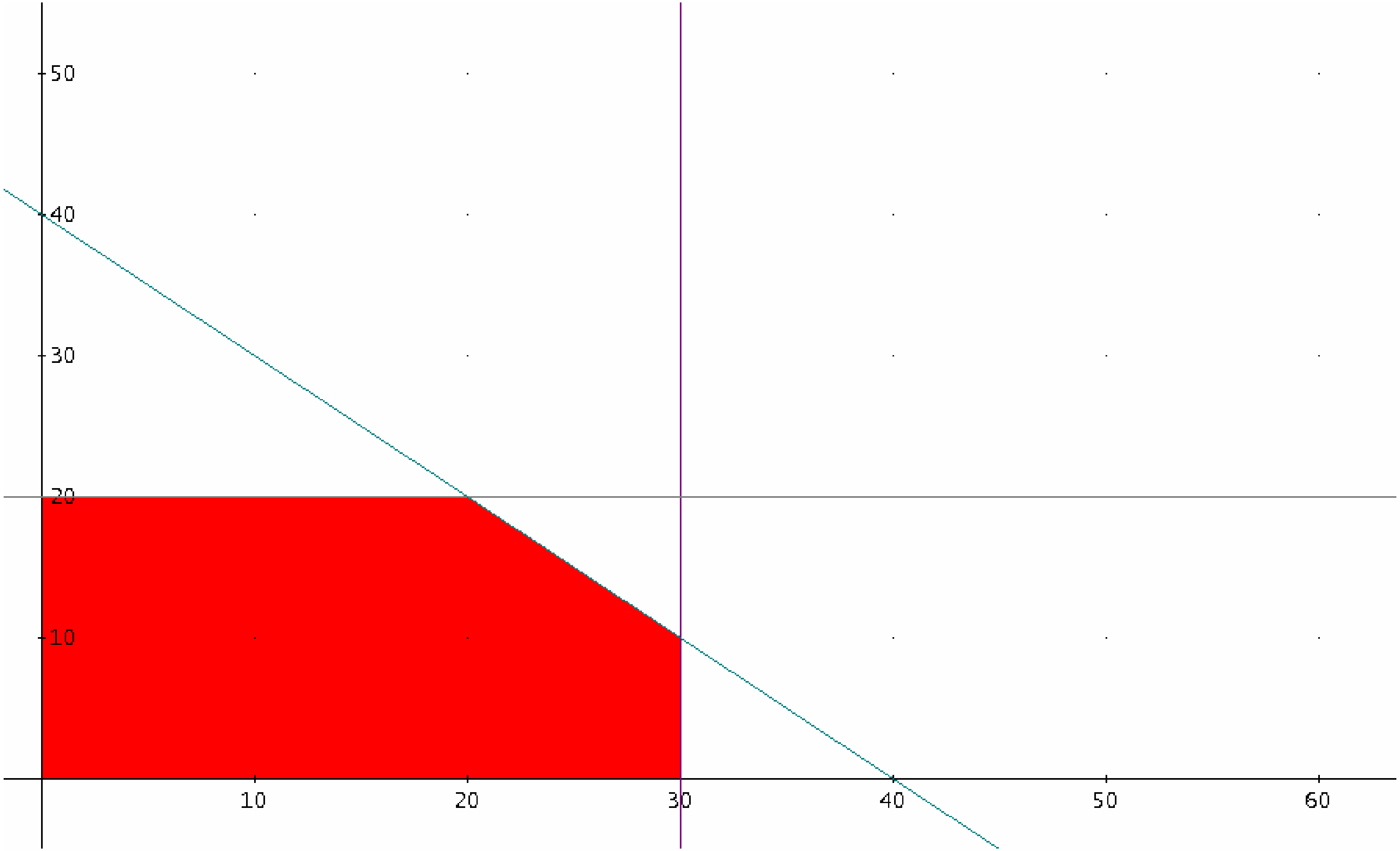
$$\max \Pi = (100 - a)a + (150 - 1.5b)b$$

$$s.t.$$

$$[cap\_1] \quad a \leq 30$$

$$[cap\_2] \quad b \leq 20$$

$$[forestcap] \quad a + b \leq 40$$


50
40
30
20
10
10
20
30
40
50
60

$$\max \Pi = (100 - a)a + (150 - 1.5b)b$$

$$s.t.$$

$$[cap\_1] \qquad a \qquad \leq 30$$

$$[cap\_2] \qquad b \leq 20$$

$$[forestcap] \quad a + b \leq 40$$

$$L = 100a - a^2 + 150b - 1.5b^2 +$$
$$+ \lambda_1(30 - a) + \lambda_2(20 - b) + \lambda_3(40 - a - b)$$

$$\frac{\partial L}{\partial \lambda_1} = 30 - a$$

$$\frac{\partial L}{\partial \lambda_2} = 20 - b$$

$$\frac{\partial L}{\partial \lambda_3} = 40 - a - b$$

$$\frac{\partial L}{\partial a} = 100 - 2a - \lambda_1 - \lambda_3$$

$$\frac{\partial L}{\partial b} = 150 - 3b - \lambda_2 - \lambda_3$$

$$\frac{\partial \Pi}{\partial a} = 100 - 2a$$

$$\frac{\partial \Pi}{\partial b} = 150 - 3b$$

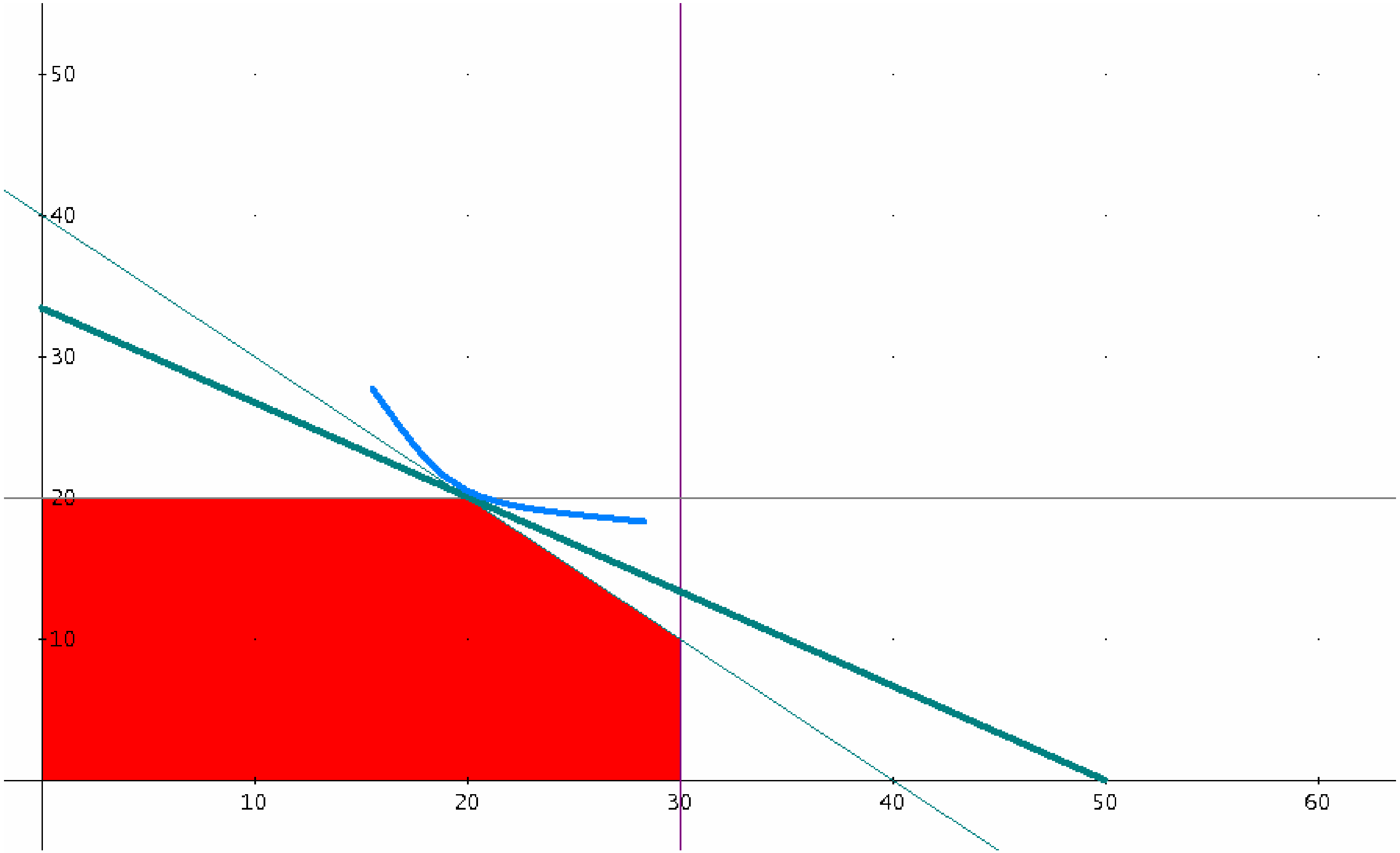
$$d\Pi = \frac{\partial \Pi}{\partial a} da + \frac{\partial \Pi}{\partial b} db = 0$$

$$\frac{\partial \Pi}{\partial b} db = -\frac{\partial \Pi}{\partial a} da$$

$$\frac{db}{da} = \frac{-\left(\frac{\partial \Pi}{\partial a}\right)}{\left(\frac{\partial \Pi}{\partial b}\right)}$$

$$\frac{db}{da} = \frac{-100 + 2a}{150 - 3b}$$

$$\frac{db}{da} = \frac{-100 + 2(20)}{150 - 3(20)} = \frac{-60}{90} = \frac{-2}{3}$$


50
40
30
20
10
10
20
30
40
50
60

Solution via Lingo:

```
model:

max = prof;

prof = (100-a)*a + (150-1.5*b)*b;

[cap_1] a <= 30;
[cap_2] b <= 20;
[forcap]a+b <= 40;

end
```

```
  Local optimal solution found.
  Objective value:                              4000.000
  Infeasibilities:                              0.000000
  Extended solver steps:                               5
  Total solver iterations:                            54
```

Variable	Value	Reduced Cost
PROF	4000.000	0.000000
A	20.00000	0.000000
B	20.00000	0.000000

Row	Slack or Surplus	Dual Price
1	4000.000	1.000000
2	0.000000	1.000000
CAP_1	10.00000	0.000000
CAP_2	0.000000	30.00000
FORCAP	0.000000	60.00000

Quadratic programming

Simplex Table =

a	b	lam1	lam2	lam3	s1	s2	s3	w1	w2	RHS
−100.0	−150.0	−30.0	−20.0	−40.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	30.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	40.0
−2.0	0.0	−1.0	0.0	−1.0	0.0	0.0	0.0	1.0	0.0	−100.0
0.0	−3.0	0.0	−1.0	−1.0	0.0	0.0	0.0	0.0	1.0	−150.0

$$\frac{\partial L}{\partial \lambda_1} = 30 \quad - a$$

$$\frac{\partial L}{\partial \lambda_2} = 20 \qquad - b$$

$$\frac{\partial L}{\partial \lambda_3} = 40 \quad - a \ - b$$

$$\frac{\partial L}{\partial a} = 100 - 2a \qquad - \lambda_1 \qquad - \lambda_3$$

$$\frac{\partial L}{\partial b} = 150 \qquad - 3b \qquad - \lambda_2 - \lambda_3$$

Detailed tables

Simple Simplex
Simpsim1.bas
Peter Lohmander 090922

Simplex Table =

a	b	lam1	lam2	lam3	s1	s2	s3	w1	w2	RHS
-100.0	-150.0	-30.0	-20.0	-40.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	30.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	40.0
-2.0	0.0	-1.0	0.0	-1.0	0.0	0.0	0.0	1.0	0.0	-100.0
0.0	-3.0	0.0	-1.0	-1.0	0.0	0.0	0.0	0.0	1.0	-150.0

a	0
b	0
lam1	0
lam2	0
lam3	0
s1	30
s2	20
s3	40
w1	-100
w2	-150
OBJECTIVE	0

Column = 2
Row = 3
Value = 1

Simplex Table =

a	b	lam1	lam2	lam3	s1	s2	s3	w1	w2	RHS
−100.0	0.0	−30.0	−20.0	−40.0	0.0	150.0	0.0	0.0	0.0	3000.0
1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	30.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	20.0
−2.0	0.0	−1.0	0.0	−1.0	0.0	0.0	0.0	1.0	0.0	−100.0
0.0	0.0	0.0	−1.0	−1.0	0.0	3.0	0.0	0.0	1.0	−90.0

a	0
b	20
lam1	0
lam2	0
lam3	0
s1	30
s2	0 Since s2=0, lam2 has to be determined by the equations!
s3	20
w1	
w2	
OBJECTIVE	

Column = 4
Row = 6
Value = −1

```
Simplex Table =

      a       b    lam1    lam2    lam3      s1      s2      s3      w1      w2     RHS

 -100.0     0.0   -30.0     0.0   -20.0     0.0    90.0     0.0     0.0   -20.0  4800.0
    1.0     0.0     0.0     0.0     0.0     1.0     0.0     0.0     0.0     0.0    30.0
    0.0     1.0     0.0     0.0     0.0     0.0     1.0     0.0     0.0     0.0    20.0
    1.0     0.0     0.0     0.0     0.0     0.0    -1.0     1.0     0.0     0.0    20.0
   -2.0     0.0    -1.0     0.0    -1.0     0.0     0.0     0.0     1.0     0.0  -100.0
    0.0     0.0     0.0     1.0     1.0     0.0    -3.0     0.0     0.0    -1.0    90.0
```

a	0
b	20
lam1	0
lam2	90
lam3	0
s1	30
s2	0
s3	20
w1	-100
w2	0
OBJECTIVE	4800/2 = 2400

```
Column =  1
Row =  4
Value =  1
```

Simplex Table =

a	b	lam1	lam2	lam3	s1	s2	s3	w1	w2	RHS
0.0	0.0	−30.0	0.0	−20.0	0.0	−10.0	100.0	0.0	−20.0	6800.0
0.0	0.0	0.0	0.0	0.0	1.0	1.0	−1.0	0.0	0.0	10.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	20.0
0.0	0.0	−1.0	0.0	−1.0	0.0	−2.0	2.0	1.0	0.0	−60.0
0.0	0.0	0.0	1.0	1.0	0.0	−3.0	0.0	0.0	−1.0	90.0

a	20
b	20
lam1	0
lam2	90
lam3	0
s1	10
s2	0
s3	0 Since s3=0, lam3 has to be determined by the equations!
w1	−60
w2	0
OBJECTIVE	

Column = 5
Row = 5
Value = −1

Simplex Table =

a	b	lam1	lam2	lam3	s1	s2	s3	w1	w2	RHS
0.0	0.0	-10.0	0.0	0.0	0.0	30.0	60.0	-20.0	-20.0	8000.0
0.0	0.0	0.0	0.0	0.0	1.0	1.0	-1.0	0.0	0.0	10.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	-1.0	1.0	0.0	0.0	20.0
0.0	0.0	1.0	0.0	1.0	0.0	2.0	-2.0	-1.0	0.0	60.0
0.0	0.0	-1.0	1.0	0.0	0.0	-5.0	2.0	1.0	-1.0	30.0

a	20
b	20
lam1	0
lam2	30
lam3	60
s1	10
s2	0
s3	0
w1	0
w2	0
OBJECTIVE	8000/2 = 4000

Programming results and software

```
Simple Simplex
Simpsim1.bas
Peter Lohmander 090922

Simplex Table =

 -100.0 -150.0  -30.0  -20.0  -40.0    0.0    0.0    0.0    0.0    0.0    0.0
    1.0    0.0    0.0    0.0    0.0    1.0    0.0    0.0    0.0    0.0   30.0
    0.0    1.0    0.0    0.0    0.0    0.0    1.0    0.0    0.0    0.0   20.0
    1.0    1.0    0.0    0.0    0.0    0.0    0.0    1.0    0.0    0.0   40.0
   -2.0    0.0   -1.0    0.0   -1.0    0.0    0.0    0.0    1.0    0.0 -100.0
    0.0   -3.0    0.0   -1.0   -1.0    0.0    0.0    0.0    0.0    1.0 -150.0
Column =  2
Row =  3
Value =  1

Simplex Table =

 -100.0    0.0  -30.0  -20.0  -40.0    0.0  150.0    0.0    0.0    0.0 3000.0
    1.0    0.0    0.0    0.0    0.0    1.0    0.0    0.0    0.0    0.0   30.0
    0.0    1.0    0.0    0.0    0.0    0.0    1.0    0.0    0.0    0.0   20.0
    1.0    0.0    0.0    0.0    0.0    0.0   -1.0    1.0    0.0    0.0   20.0
   -2.0    0.0   -1.0    0.0   -1.0    0.0    0.0    0.0    1.0    0.0 -100.0
    0.0    0.0    0.0   -1.0   -1.0    0.0    3.0    0.0    0.0    1.0  -90.0
Column =  4
Row =  6
Value = -1
```

Simplex Table =

−100.0	0.0	−30.0	0.0	−20.0	0.0	90.0	0.0	0.0	−20.0	4800.0
1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	30.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	20.0
−2.0	0.0	−1.0	0.0	−1.0	0.0	0.0	0.0	1.0	0.0	−100.0
0.0	0.0	0.0	1.0	1.0	0.0	−3.0	0.0	0.0	−1.0	90.0

Column = 1
Row = 4
Value = 1

Simplex Table =

0.0	0.0	−30.0	0.0	−20.0	0.0	−10.0	100.0	0.0	−20.0	6800.0
0.0	0.0	0.0	0.0	0.0	1.0	1.0	−1.0	0.0	0.0	10.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	20.0
0.0	0.0	−1.0	0.0	−1.0	0.0	−2.0	2.0	1.0	0.0	−60.0
0.0	0.0	0.0	1.0	1.0	0.0	−3.0	0.0	0.0	−1.0	90.0

Column = 5
Row = 5
Value = −1

Simplex Table =

0.0	0.0	−10.0	0.0	0.0	0.0	30.0	60.0	−20.0	−20.0	8000.0
0.0	0.0	0.0	0.0	0.0	1.0	1.0	−1.0	0.0	0.0	10.0
0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	20.0
1.0	0.0	0.0	0.0	0.0	0.0	−1.0	1.0	0.0	0.0	20.0
0.0	0.0	1.0	0.0	1.0	0.0	2.0	−2.0	−1.0	0.0	60.0
0.0	0.0	−1.0	1.0	0.0	0.0	−5.0	2.0	1.0	−1.0	30.0

```
REM
CLS
OPEN ”outsimp.txt” FOR OUTPUT AS #1
PRINT ”Simple Simplex”
PRINT ”Simpsim1.bas”
PRINT ”Peter Lohmander 090922”
PRINT #1, ”Simple Simplex”
PRINT #1, ”Simpsim1.bas”
PRINT #1, ”Peter Lohmander 090922”

REM

mmax = 6
nmax = 11
DIM A(11, 11)

FOR m = 1 TO mmax
FOR n = 1 TO nmax
A(m, n) = 0
NEXT n
NEXT m

REM
REM Initial simplex table
REM

A(1, 1) = -100
A(1, 2) = -150
A(1, 3) = -30
A(1, 4) = -20
A(1, 5) = -40
A(2, 1) = 1
```

```
A(2, 6) = 1
A(2, 11) = 30
A(3, 2) = 1
A(3, 7) = 1
A(3, 11) = 20
A(4, 1) = 1
A(4, 2) = 1
A(4, 8) = 1
A(4, 11) = 40
A(5, 1) = −2
A(5, 3) = −1
A(5, 5) = −1
A(5, 9) = 1
A(5, 11) = −100
A(6, 2) = −3
A(6, 4) = −1
A(6, 5) = −1
A(6, 10) = 1
A(6, 11) = −150

1 REM
PRINT ””
PRINT ”Simplex Table = ”
PRINT ””
PRINT #1, ””
PRINT #1, ”Simplex Table = ”
PRINT #1, ””
```

```
FOR m = 1 TO mmax
FOR n = 1 TO nmax
PRINT USING "#####.#"; A(m, n);
PRINT #1, USING "#####.#"; A(m, n);
NEXT n
PRINT ""
PRINT #1, ""
NEXT m

PRINT ""
INPUT "Have you reached the final table (No = 0, Yes = 1) ", slut
IF slut = 1 THEN GOTO 55
INPUT "Please select column!", L
PRINT "Column = "; L
PRINT #1, "Column = "; L

INPUT "Please select row!", k
PRINT "Row = "; k
PRINT #1, "Row = "; k
divi = A(k, L)
PRINT "Value = "; divi
PRINT #1, "Value = "; divi

FOR n = 1 TO nmax
A(k, n) = A(k, n) / divi
NEXT n

FOR m = 1 TO mmax
IF m = k THEN 10
TAL = A(m, L)

FOR n = 1 TO nmax
A(m, n) = A(m, n) - TAL * A(k, n)
NEXT n
```

```
10 REM
NEXT m

GOTO 1

55 REM
CLOSE #1

END
```