L3 (By Peter Lohmander 2009-09-16)

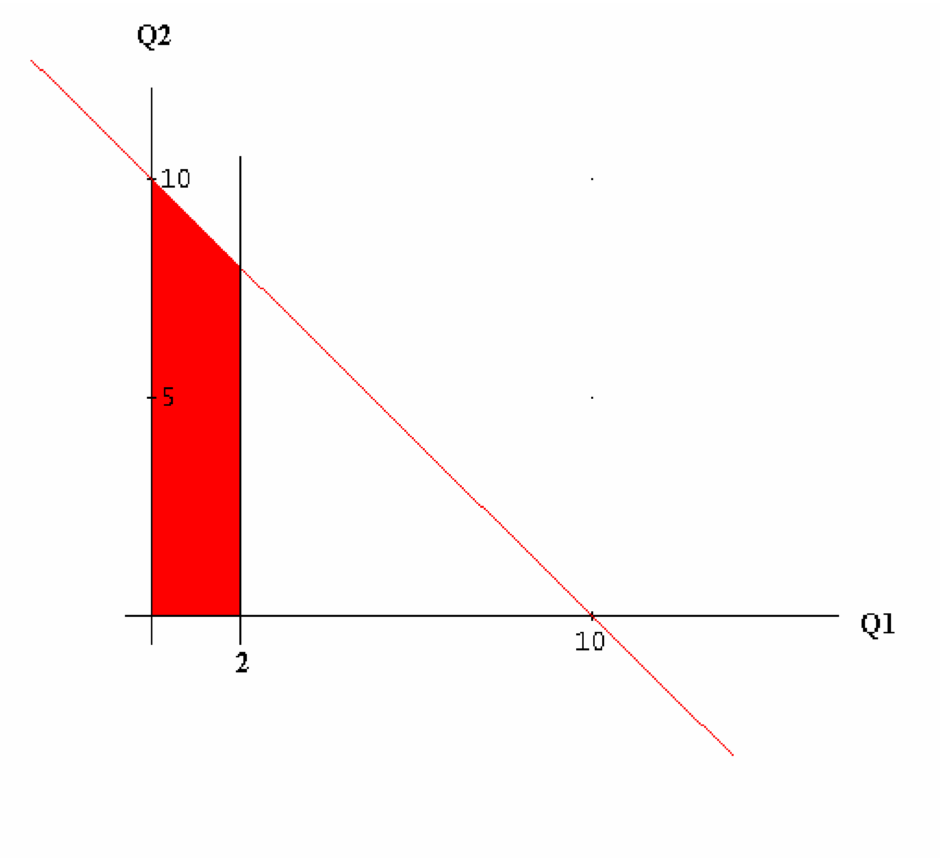
Linear programming with binary and integer constraints. General theory, application examples, analytical solutions and numerical solutions via computer programming.
Search for optimal solutions in problems or low dimensionality where you can not solve the optimization problem via a linear equation system of first order conditions. Low level programming and application of fundamental numerical methods. Introduction to low level programming software.

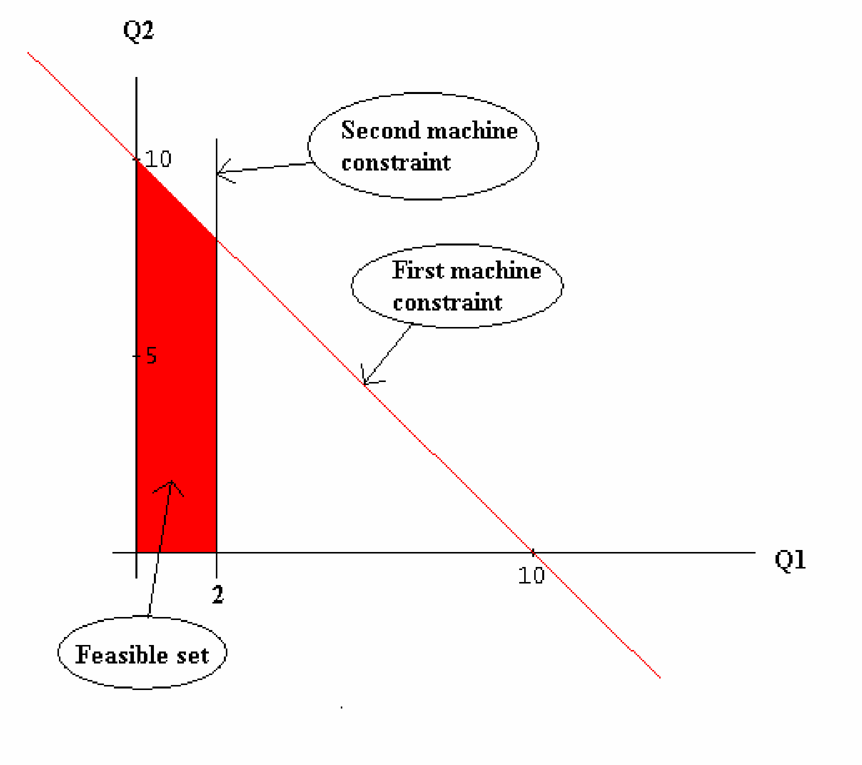
Numerical methods to solve linear programming problems

The background to the numerical approach is found in the course book.

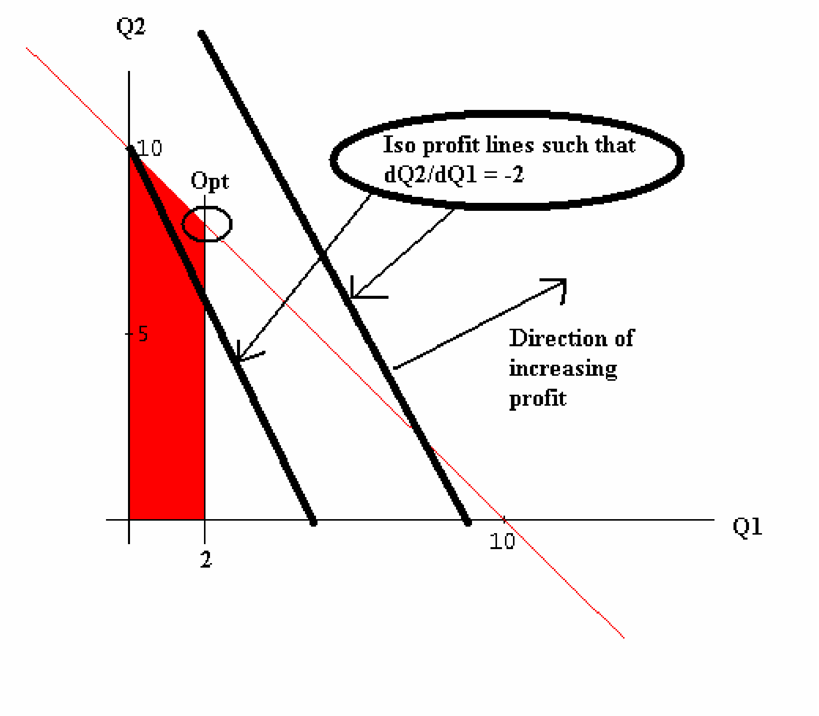
Example 1:
The simplex method applied to LP problem with "<=" constraints.

$$\max \Pi = 10q_1 + 5q_2$$
$$s.t.$$
$$1q_1 + 1q_2 \le 10$$
$$1q_1 \le 2$$





Q2
10
Second machine constraint
First machine constraint
5
Q1
2
10
Feasible set



Q2
10
Opt
Iso profit lines such that dQ2/dQ1 = -2
5
Direction of increasing profit
Q1
2
10

Table 1.

Q1	*Q2*	*S1*	*S2*	*RHS*
-10	*-5*			
1	*1*	*1*		*10*
1			*1*	*2*

Q1 = 0 Reduced cost 1 = -10
Q2 = 0 Reduced cost 2 = -5
S1 = 10 Shadow price 1 = 0
S2 = 2 Shadow price 2 = 0

Objective function = 0

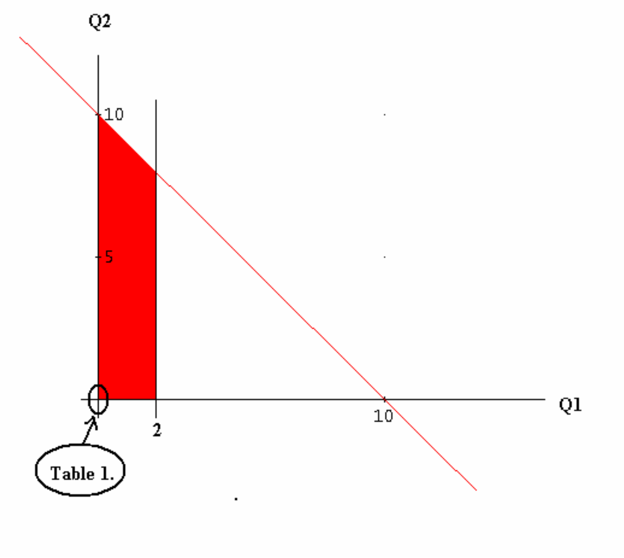


Table 2.

Q1	*Q2*	*S1*	*S2*	*RHS*
0	*-5*		*10*	*20*
0	*1*	*1*	*-1*	*8*
1			*1*	*2*

Q1 = 2 Reduced cost 1 = 0
Q2 = 0 Reduced cost 2 = -5
S1 = 8 Shadow price 1 = 0
S2 = 0 Shadow price 2 = 10

Objective function = 20

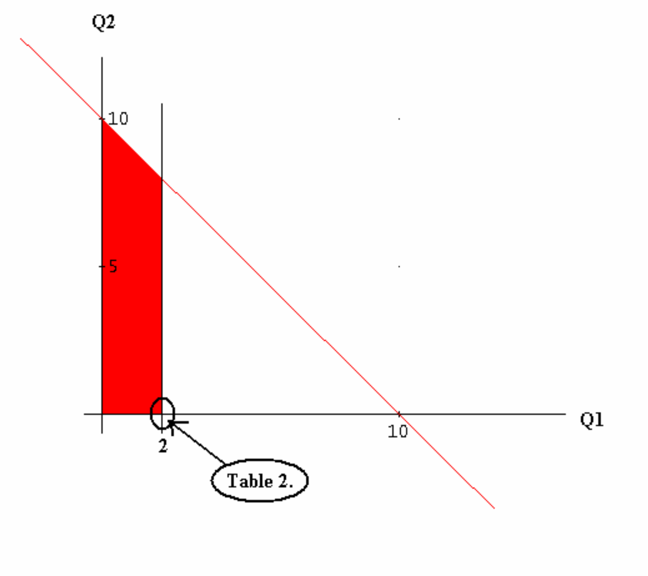
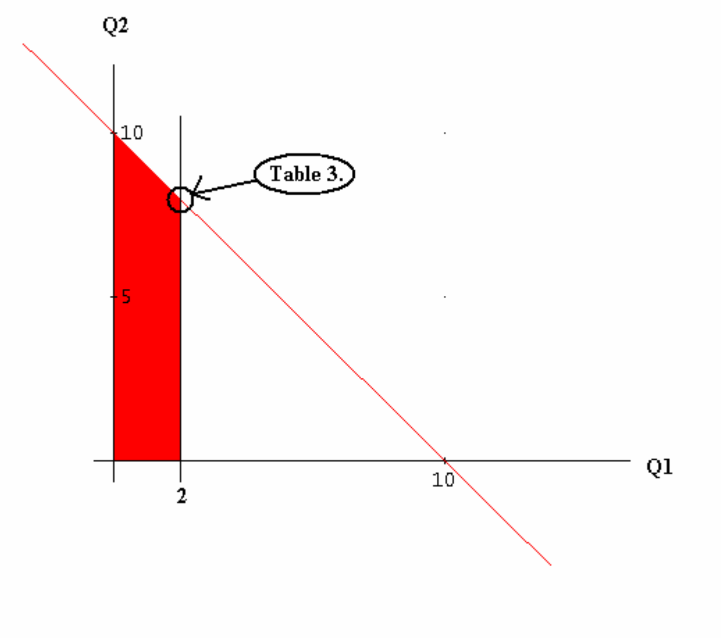


Table 3.

Q1	*Q2*	*S1*	*S2*	*RHS*
0	*0*	*5*	*5*	*60*
0	*1*	*1*	*-1*	*8*
1			*1*	*2*

Q1 = 2 Reduced cost 1 = 0
Q2 = 8 Reduced cost 2 = 0
S1 = 0 Shadow price 1 = 5
S2 = 0 Shadow price 2 = 5

Objective function = 60



Primal problem:

$$\max \Pi = 10q_1 + 5q_2$$

$s.t.$

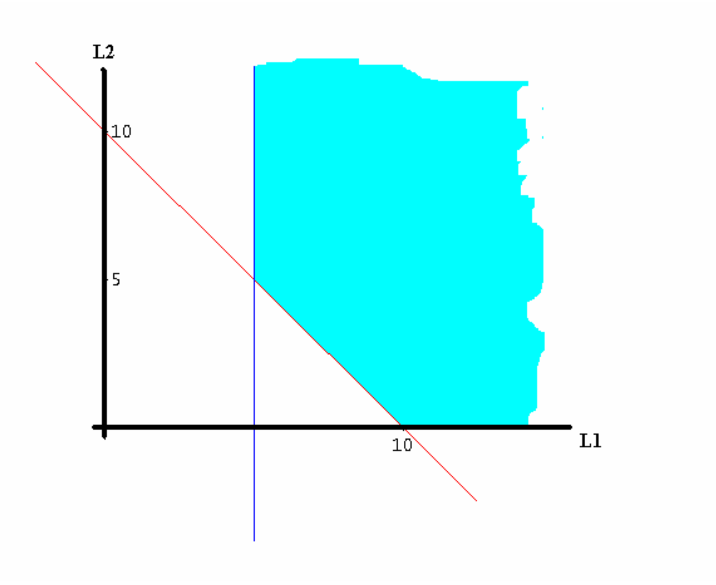
$$1q_1 + 1q_2 \le 10$$
$$1q_1 \le 2$$

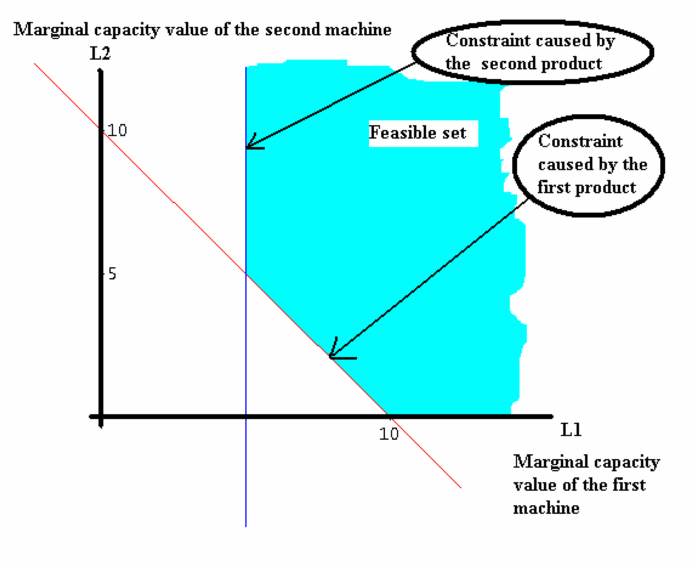
Dual problem:

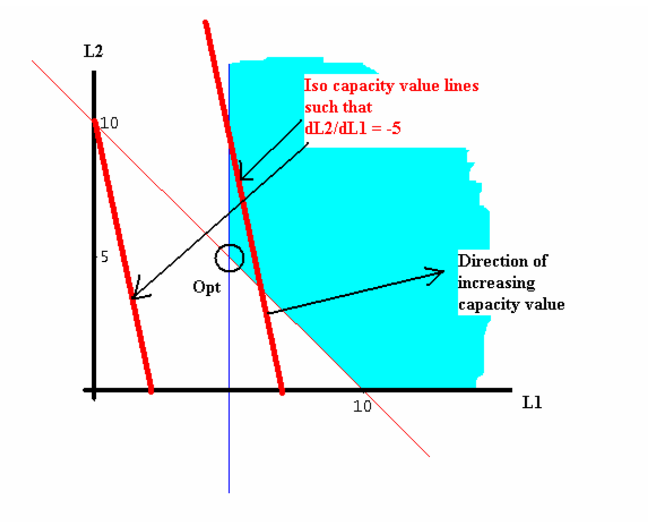
$$\min \overline{\Pi} = 10\lambda_1 + 2\lambda_2$$

$s.t.$

$$1\lambda_1 + 1\lambda_2 \ge 10$$
$$1\lambda_1 \ge 5$$


L2
10
5
10
L1


Marginal capacity value of the second machine
L2
Constraint caused by the second product
10
Feasible set
Constraint caused by the first product
5
10
L1
Marginal capacity value of the first machine


L2
Iso capacity value lines such that dL2/dL1 = -5
10
5
Opt
Direction of increasing capacity value
10
L1

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 10;

[M2] 1*q1 <= 2;

end
```

```
Global optimal solution found.
  Objective value:                              60.00000
  Infeasibilities:                              0.000000
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        60.00000            0.000000
                             Q1        2.000000            0.000000
                             Q2        8.000000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        60.00000            1.000000
                              2        0.000000            1.000000
                             M1        0.000000            5.000000
                             M2        0.000000            5.000000


 Ranges in which the basis is unchanged:

                                  Objective Coefficient Ranges
                              Current        Allowable        Allowable
                 Variable   Coefficient        Increase         Decrease
                     PROF      1.000000        INFINITY         1.000000
                       Q1           0.0        INFINITY         5.000000
                       Q2           0.0        5.000000         5.000000

                                       Righthand Side Ranges
                      Row       Current        Allowable        Allowable
                                    RHS        Increase         Decrease
                        2           0.0        INFINITY         60.00000
                       M1      10.00000        INFINITY         8.000000
                       M2      2.000000        8.000000         2.000000
```

Example 2:

Primal problem:

$$\max \Pi = 40q_1 + 60q_2$$
$$s.t.$$
$$4q_1 + 8q_2 \leq 1000$$
$$10q + 5q_2 \leq 1000$$

40 q1	60 q2			
4 q1	8 q2	1 s1		1000
10 q1	5 q2		1 s2	1000

Table 1.

-40	-60			
4	8	1		1000
10	5		1	1000

B enters and the element 8 is the found in the most constraining constraint. We divide every element in that constraint by 8 and get this new constraint row:

1/2	1	1/8		125

Table 2.

-10	0	15/2		7500
1/2	1	1/8		125
15/2	0	-5/8	1	375

A enters and the element 15/2 is the found in the most constraining constraint. We divide every element in that constraint by 15/2 and get this new constraint row:

1	0	-1/12	2/15	50

Table 3.

0	0	20/3	4/3	8000
0	1	1/6	-1/15	100
1	0	-1/12	2/15	50

We may investigate the dual problem and make sure that the solution is really optimal. Of course, it is a good idea also to draw the primal and dual graphs.

```
model:

max = prof;

prof = 40*q1 + 60*q2;

[M1] 4*q1 + 8*q2 <= 1000;

[M2] 10*q1 + 5*q2 <= 1000;

end
```

```
Global optimal solution found.
  Objective value:                              8000.000
  Infeasibilities:                              0.000000
  Total solver iterations:                             2


                       Variable           Value        Reduced Cost
                           PROF        8000.000            0.000000
                             Q1        50.00000            0.000000
                             Q2        100.0000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        8000.000            1.000000
                              2        0.000000            1.000000
                             M1        0.000000            6.666667
                             M2        0.000000            1.333333
```

```
Ranges in which the basis is unchanged:

                                   Objective Coefficient Ranges
                               Current        Allowable        Allowable
                 Variable  Coefficient         Increase         Decrease
                     PROF     1.000000         INFINITY         1.000000
                       Q1          0.0         80.00000         10.00000
                       Q2          0.0         20.00000         40.00000

                                       Righthand Side Ranges
                      Row      Current        Allowable        Allowable
                                   RHS         Increase         Decrease
                        2          0.0         INFINITY         8000.000
                       M1     1000.000         600.0000         600.0000
                       M2     1000.000         1500.000         375.0000
```

Example 3:

Primal problem:

$$\max \Pi = 2q_1 + 1q_2$$
$$s.t.$$
$$1q_1 + 1q_2 \le 10$$
$$1q_2 \ge 5$$

Here, we need a surplus variable, s2, and an artificial variable, y1. We let "BigM" = 1000.

2 q1	1 q1			-1000 y1	
1 q2	1 q2	1 s1			10
	1 q2		- 1 s2	1 y1	5

Table 1.

- 2	- 1			1000	
1	1	1			10
	1		- 1	1	5

The first step is to make sure that the artificial variables(s) enter the solution.

Table 2.

- 2	- 1001		1000		-5000
1	1	1			10
	1		- 1	1	5

Then, we continue the process as usual. q2 should enter the solution.

Table 3.

- 2	0		-1	1001	5
1	0	1	1	-1	5
	1		- 1	1	5

We continue the process as usual. q1 should enter the solution.

Table 4.

0	0	2	1	999	15
1	0	1	1	-1	5
	1		- 1	1	5

It is a good idea to draw the primal graph and check the solution.

```
model:

max = prof;

prof = 2*q1 + 1*q2;

[M1] 1*q1 + 1*q2 <= 10;

[GRT] 1*q2 >= 5;

end
```

```
  Global optimal solution found.
  Objective value:                              15.00000
  Infeasibilities:                              0.000000
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        15.00000            0.000000
                             Q1        5.000000            0.000000
                             Q2        5.000000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        15.00000            1.000000
                              2        0.000000            1.000000
                             M1        0.000000            2.000000
                            GRT        0.000000           -1.000000


Ranges in which the basis is unchanged:

                                  Objective Coefficient Ranges
                              Current        Allowable        Allowable
                Variable    Coefficient       Increase         Decrease
                    PROF       1.000000       INFINITY         1.000000
                      Q1            0.0       INFINITY         1.000000
                      Q2            0.0       1.000000         INFINITY

                                      Righthand Side Ranges
                     Row        Current        Allowable        Allowable
                                    RHS         Increase         Decrease
                       2            0.0       INFINITY         15.00000
                      M1       10.00000       INFINITY         5.000000
                     GRT       5.000000       5.000000         5.000000
```

Example 4:

First, we recall this problem:

$$\max \Pi = 10q_1 + 5q_2$$
$$s.t.$$
$$1q_1 + 1q_2 \leq 10$$
$$1q_1 \leq 2$$

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 10;

[M2] 1*q1 <= 2;

end
```

```
  Global optimal solution found.
  Objective value:                              60.00000
  Infeasibilities:                              0.000000
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        60.00000            0.000000
                             Q1        2.000000            0.000000
                             Q2        8.000000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        60.00000            1.000000
                              2        0.000000            1.000000
                             M1        0.000000            5.000000
                             M2        0.000000            5.000000
```

Then, we modify it marginally:

$$\max \Pi = 10q_1 + 5q_2$$

$s.t.$

$$1q_1 + 1q_2 \leq 9.9$$

$$1q_1 \leq 2$$

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 9.9;

[M2] 1*q1 <= 2;

end
```

```
  Global optimal solution found.
  Objective value:                              59.50000
  Infeasibilities:                              0.000000
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        59.50000            0.000000
                             Q1        2.000000            0.000000
                             Q2        7.900000            0.000000

                            Row    Slack or Surplus      Dual Price
                              1        59.50000            1.000000
                              2        0.000000            1.000000
                             M1        0.000000            5.000000
                             M2        0.000000            5.000000
```

Then, we demand a "general integer" solution from the modified problem:

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 9.9;

[M2] 1*q1 <= 2;

@GIN(q1);
@GIN(q2);

end
```

```
  Global optimal solution found.
  Objective value:                              55.00000
  Objective bound:                              55.00000
  Infeasibilities:                              0.000000
  Extended solver steps:                               0
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        55.00000            0.000000
                             Q1        2.000000           -10.00000
                             Q2        7.000000           -5.000000

                            Row    Slack or Surplus      Dual Price
                              1        55.00000            1.000000
                              2        0.000000            1.000000
                             M1       0.9000000            0.000000
                             M2        0.000000            0.000000
```

We could also have demanded that q2 is a binary variable:

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 9.9;

[M2] 1*q1 <= 2;

@GIN(q1);
@BIN(q2);

end
```

```
  Global optimal solution found.
  Objective value:                              25.00000
  Objective bound:                              25.00000
  Infeasibilities:                              0.000000
  Extended solver steps:                               0
  Total solver iterations:                             0


                       Variable           Value        Reduced Cost
                           PROF        25.00000            0.000000
                             Q1        2.000000           -10.00000
                             Q2        1.000000           -5.000000

                            Row    Slack or Surplus      Dual Price
                              1        25.00000            1.000000
                              2        0.000000            1.000000
                             M1        6.900000            0.000000
                             M2        0.000000            0.000000
```

Example 5.
Automatic generation of the dual problem

Start with this primal problem:

```
model:

max = prof;

prof = 10*q1 + 5*q2;

[M1] 1*q1 + 1*q2 <= 10;

[M2] 1*q1 <= 2;

end
```

Then, click on:
Lingo, Generate, Dual model.

Then, you get:

```
MODEL:
MIN = 10 * M1 + 2 * M2;
[ PROF]  _2 >= 1;
[ Q1] - 10 * _2 +  M1 +  M2 >= 0;
[ Q2] - 5 * _2 +  M1 >= 0;
@FREE( _2);
END
```