Attrition coefficient estimations

via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions

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ABSTRACT:

In battles with aimed fire, the attrition of a force can under simplified assumptions be shown to be proportional to the number of enemies. Lanchester models for aimed fire are differential equation systems that can be applied to describe the dynamics of such battles. In order to determine the attrition coefficients and the complete dynamics of the battle in continuous time, the following procedure is introduced: First, the general solution of the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived. The four parameters of the solution are determined. In these equations, the initial and terminal sizes of the two forces, are parameters. A 4-dimensional fix point iteration algorithm is developed and implemented as a computer code, that rapidly solves the nonlinear equation system. After 40 iterations, the absolute relative errors in all equations are smaller than 10^(-12) . Then, a discrete time version of the Lanchester differential equation system, with stochastic attrition coefficients, is defined as a difference equation system. The effects of increasing risk in the attrition coefficients, that determine how the time derivative of the size of force X is affected by the size of force Y, at different points in time, is analyzed. It is shown that the expected size of force X is a strictly convex function of the risk in the attrition coefficients. According to the Jensen's inequality, the expected size of force X at time t+2 is a strictly increasing function of the risk in the attrition coefficients at time t and t+1 for arbitrary values of t. In case the attrition coefficients in different periods are stochastic, and the system parameters are determined according to the suggested procedure, then the expected attrition coefficients obtain higher values than if the attrition coefficients would be constant over time. This can explain differences between attrition coefficient estimates based on different methods and coefficient risk assumptions.

Abstract Part 1:

- In battles with aimed fire, the attrition of a force can under simplified assumptions be shown to be proportional to the number of enemies. Lanchester models for aimed fire are differential equation systems that can be applied to describe the dynamics of such battles.
- In order to determine the attrition coefficients and the complete dynamics of the battle in continuous time, the following procedure is introduced:
- First, the general solution of the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived.

Abstract Part 2:

- The four parameters of the solution are determined. In these equations, the initial and terminal sizes of the two forces, are parameters.
- A 4-dimensional fix point iteration algorithm is developed and implemented as a computer code, that rapidly solves the nonlinear equation system. After 40 iterations, the absolute relative errors in all equations are smaller than 10^{-12}.

Abstract Part 3:

- Then, a discrete time version of the Lanchester differential equation system, with stochastic attrition coefficients, is defined as a difference equation system.
- The effects of increasing risk in the attrition coefficients, that determine how the time derivative of the size of force X is affected by the size of force Y, at different points in time, is analyzed.

Abstract Part 4:

- It is shown that the expected size of force X is a strictly convex function of the risk in the attrition coefficients.
- According to the Jensen's inequality, the expected size of force X at time t+2 is a strictly increasing function of the risk in the attrition coefficients at time t and t+1 for arbitrary values of t.
- In case the attrition coefficients in different periods are stochastic, and the system parameters are determined according to the suggested procedure, then the expected attrition coefficients obtain higher values than if the attrition coefficients would be constant over time.
- This can explain differences between attrition coefficient estimates based on different methods and coefficient risk assumptions.

Empirical data:

The Battle of Iwo Jima during World War II:

The Battle of Iwo Jima:

(AI generated picture)

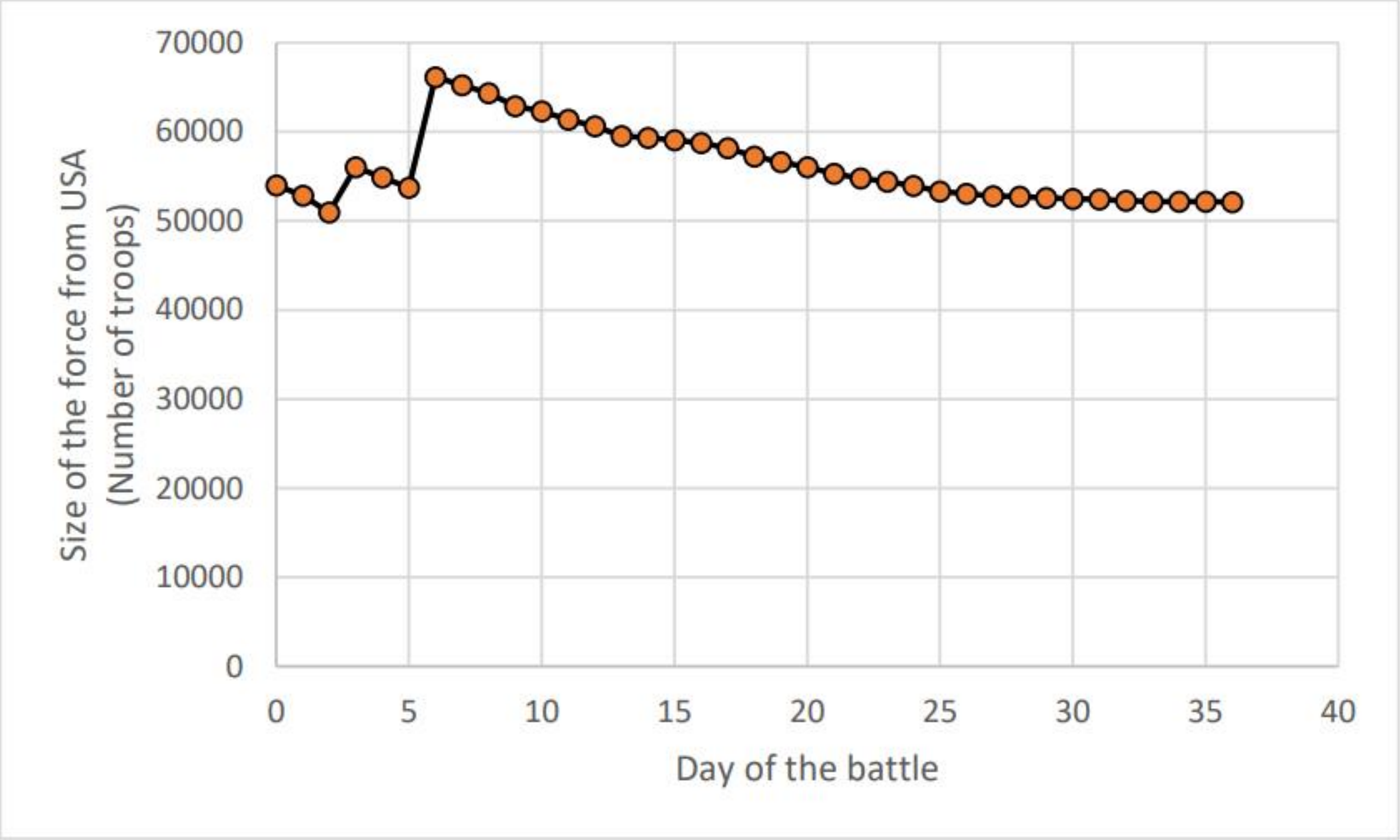


Figure 1.

The dynamically changing size of force X, the force from USA, in the battle of Iwo Jima.
Source: Data reported in Stymfal (2022), based on Engel (1954).

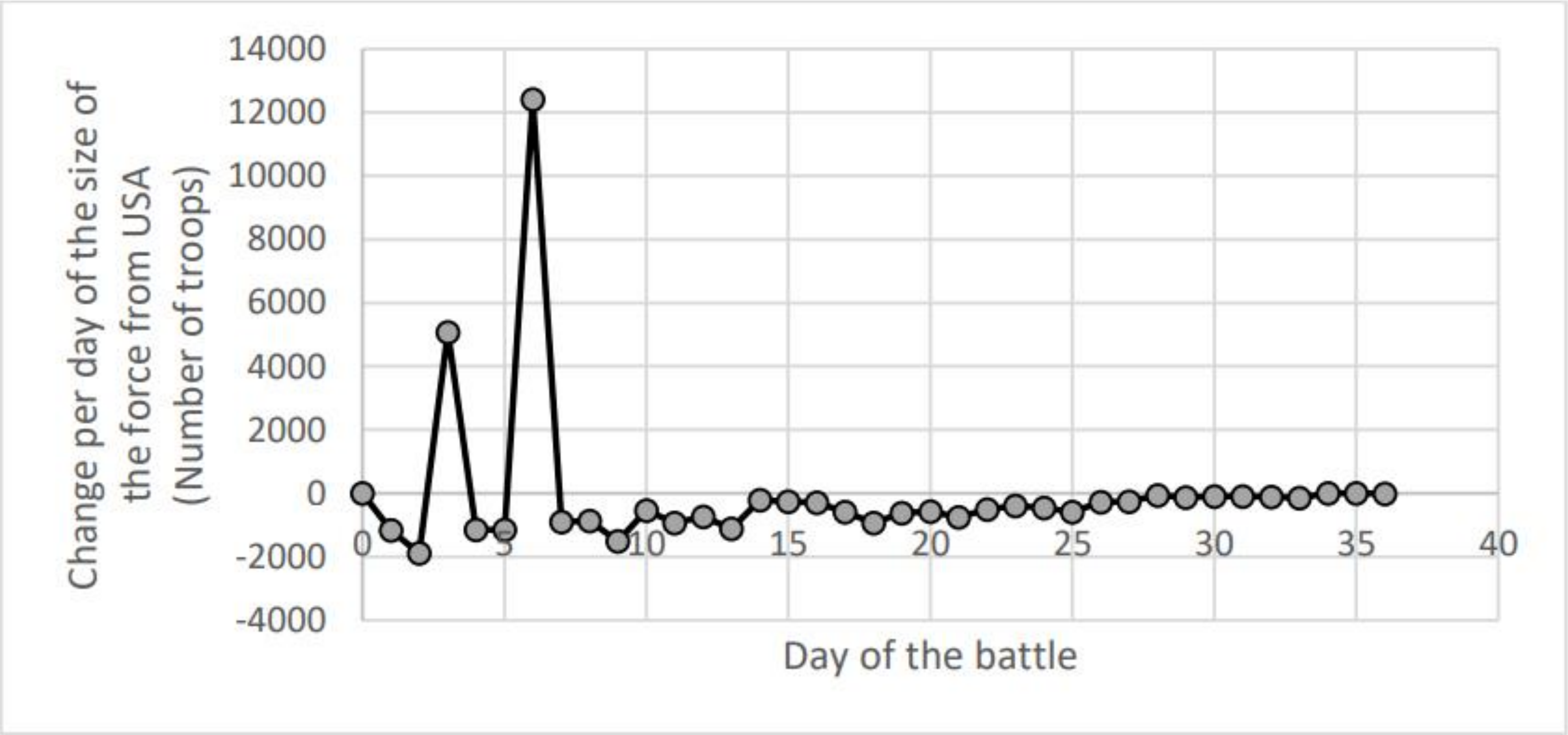


Figure 2.

The change per day, the difference, of the size of force X, the force from USA, in the battle of Iwo Jima. The graph shows the differences from Day 0 until the end of the battle. The force difference during Day t, is defined as the size of the force during Day t minus the size of the force during Day t-1. (In the graph, the size of the force in Day -1 was assumed to be identical to the size of the force in Day 0.) Reinforcements took place Day 3 and Day 6. Source: Derivations based on the data reported in Stymfal (2022), based on Engel (1954).

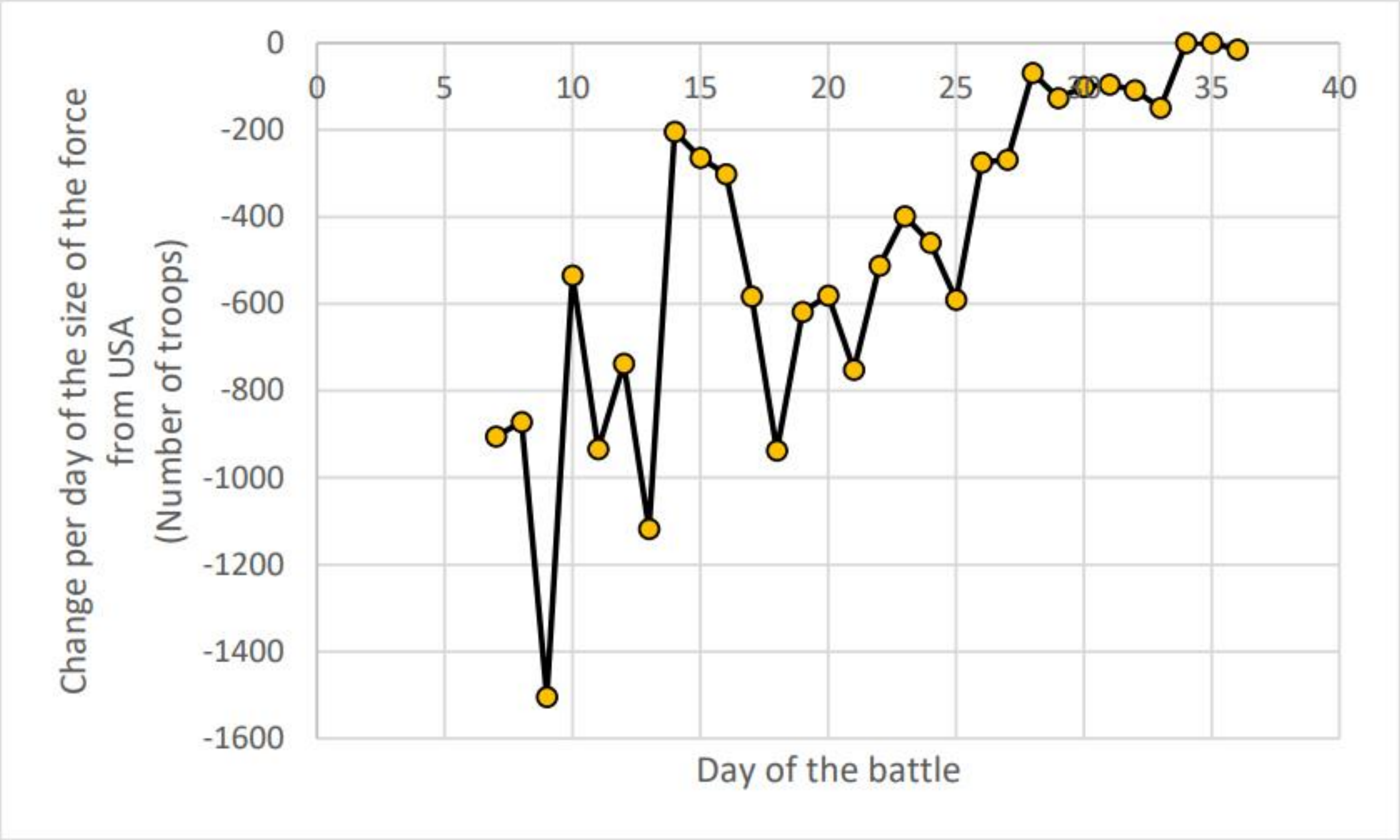


Figure 3.

The change per day, the difference, of the size of force X, the force from USA, in the battle of Iwo Jima. The graph shows the differences from Day 7 until the end of the battle. The force difference during Day t, is defined as the size of the force during Day t minus the size of the force during Day t-1. Hence, the graph is based on data covering the size of the force from Day 6 until Day 36. Source: Derivations are based on the data reported in Stymfal (2022), based on Engel (1954).

We study the differential equation system (1). There we see how the state of the system, (x, y), representing the sizes of the two opposing forces, changes over time, $t, 0 \le t \le T \ll \infty$. The two parameters, (a,b), are denoted attrition coefficients. Newtonian notation, with time derivatives marked by dots, is used.

$$\begin{cases} \dot{x} = -ay & (1.a) \\ \dot{y} = -bx & (1.b) \end{cases} \qquad a > 0, b > 0 \qquad (1)$$

From (1.a), we get (2).

$$y = -a^{-1}\dot{x} \tag{2}$$

Differentiation of (2) with respect to time, gives (3).

$$\dot{y} = -a^{-1}\ddot{x} \tag{3}$$

(3) and (1.b) give (4), which can be rewritten as (5) and (6), which is a homogenous second order differential equation.

$$-a^{-1}\ddot{x} = -bx \tag{4}$$

$$a^{-1}\ddot{x} - bx = 0 \tag{5}$$

$$\ddot{x} - abx = 0 \tag{6}$$

Let us assume that the functional form (7) is relevant. The parameters (m, λ) are assumed to be strictly different from zero.

$$x(t) = me^{\lambda t}, \quad m \neq 0, \lambda \neq 0, 0 \leq t \leq T \ll \infty \tag{7}$$

Then, the following procedure can be used to determine the state variable as an explicit function of time. Equations (6) and (7) give (8).

$$\lambda^2 me^{\lambda t} - abme^{\lambda t} = 0 \tag{8}$$

Equation (8) can be simplified to (9).

$$\left(\lambda^2 - ab\right)me^{\lambda t} = 0 \tag{9}$$

Equations (7) and (9) imply (10).

$$\lambda^2 - ab = 0 \tag{10}$$

From the quadratic equation (10), we obtain the solution (11).

$$\lambda = \pm\sqrt{ab} \tag{11}$$

Let r be defined according to (12).

$$r = \sqrt{ab} \tag{12}$$

Clearly, two solutions exist.

$$\lambda_1 = -r \tag{13}$$

$$\lambda_2 = r \tag{14}$$

<u>Observation:</u>

$a > 0 \wedge b > 0$, as we see in equation (1), which means that there are two real roots. These roots have different values. Hence, the general solution of the differential equation is:

$$x(t) = m_1 e^{-rt} + m_2 e^{rt} \tag{15}$$

Furthermore, from (2) we already know that: $y = -a^{-1}\dot{x}$

As a result, we get (16).

$$y(t) = -a^{-1}\left(-rm_1 e^{-rt} + rm_2 e^{rt}\right) \tag{16}$$

The expression (16) may be rewritten as (17).

$$y(t) = \frac{r}{a} m_1 e^{-rt} - \frac{r}{a} m_2 e^{rt} \tag{17}$$

Hence, the solution to the differential equation system (1) is given in (18).

$$\begin{cases} x(t) = & m_1 e^{-rt} + m_2 e^{rt} \\ y(t) = & \frac{r}{a} m_1 e^{-rt} - \frac{r}{a} m_2 e^{rt} \end{cases} \tag{18}$$

In order to determine the time path $\left(x(t), y(t)\right)$ we need to know the four parameters $\left(m_1, m_2, a, r\right)$.

In order to determine the time path $(x(t), y(t))$ we need to know the four parameters (m_1, m_2, a, r).

We may use four boundary conditions to determine these parameters. We already know the initial and terminal conditions of the system, namely (x_0, y_0) and (x_T, y_T).

From equation (18), the initial conditions (19) and (20) follow:

$$x(0)=m_1+m_2=x_0 \tag{19}$$

$$y(0)=\frac{r}{a}m_1-\frac{r}{a}m_2=y_0 \tag{20}$$

The terminal conditions, (21) and (22), are also derived from equation (18):

$$x(T)=m_1e^{-rT}+m_2e^{rT}=x_T \tag{21}$$

$$y(T)=\frac{r}{a}m_1e^{-rT}-\frac{r}{a}m_2e^{rT}=y_T \tag{22}$$

We may now determine the values of the parameters (m_1, m_2, a, r). The nonlinear simultaneous equation system (23) should be solved. We assume that a feasible solution exists and that this solution is unique.

$$\begin{cases} m_1 + m_2 & = x_0 & (23.a) \\ m_1 e^{-rT} + m_2 e^{rT} & = x_T & (23.b) \\ \frac{r}{a} m_1 - \frac{r}{a} m_2 & = y_0 & (23.c) \\ \frac{r}{a} m_1 e^{-rT} - \frac{r}{a} m_2 e^{rT} & = y_T & (23.d) \end{cases} \quad (23)$$

The solution of the simultaneous equation system can be found via an iteration algorithm, which is developed here:

The initial guesses of the values of the parameters are given in (24).

$$(m_1, m_2, a, r) = (m_1^0, m_2^0, a^0, r^0) \quad (24)$$

The values of (m_1, m_2, a, r) are sequentially updated. The iteration number is $i, i \in \{0,1,...,I\}$. The value of a parameter, η, after i iteration steps, is denoted η^i.

$$(Eq.23.a) \Rightarrow \left(m_1^{i+1} = x_0 - m_2^i \right) \tag{25}$$

$$(Eq.23.b) \Rightarrow \left(m_2^{i+1} = x_T e^{-r^i T} - m_1^{i+1} e^{-2r^i T} \right) \tag{26}$$

$$(Eq.23.c) \Rightarrow \left(a^{i+1} = \frac{r^i (m_1^{i+1} - m_2^{i+1})}{y_0} \right) \tag{27}$$

$$(Eq.23.d) \Rightarrow \left(\frac{r^i}{a^{i+1}} m_2^{\,i+1} e^{r^{i+1} T} = \frac{r^i}{a^{i+1}} m_1^{\,i+1} e^{-r^i T} - y_T \right) \tag{28}$$

From (28), we get (29), (30) and (31).

$$e^{r^{i+1}T} = \frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}} \tag{29}$$

$$r^{i+1}T = LN\left(\frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}}\right) \tag{30}$$

$$r^{i+1} = \frac{LN\left(\frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}}\right)}{T} \tag{31}$$

However, even if (31) is mathematically correct, it turns out that the solution to the equation system (23) sometimes diverges if equation (31) is used directly. This means that if the initial parameter guesses (24) are not quite correct, the iteration method will not make the solution approach the correct solution. Fortunately, as will be shown, it is very easy to obtain convergence in the algorithm. In the modified algorithm, the value of the parameter r changes less rapidly than if equation (31) would be used directly. The absolute change of r in the adjusted algorithm, is smaller than according to (31), but the change of r is proportional to the change that would take place if (31) would be directly applied. In equation (32), the adjustment speed parameter h is introduced.

However, even if (31) is mathematically correct, it turns out that the solution to the equation system (23) sometimes diverges if equation (31) is used directly. This means that if the initial parameter guesses (24) are not quite correct, the iteration method will not make the solution approach the correct solution. Fortunately, as will be shown, it is very easy to obtain convergence in the algorithm. In the modified algorithm, the value of the parameter r changes less rapidly than if equation (31) would be used directly. The absolute change of r in the adjusted algorithm, is smaller than according to (31), but the change of r is proportional to the change that would take place if (31) would be directly applied. In equation (32), the adjustment speed parameter h is introduced.

$$r^{i+1} = r^i + h\left(\frac{LN\left(\frac{\left(e^{-r^i T} \frac{r^i}{a^{i+1}} m_1^{i+1} - y_T \right)}{\left(\frac{r^i}{a^{i+1}} m_2^{i+1} \right)} \right)}{T} - r^i \right) \tag{32}$$

If $h = 1$, then equation (32) corresponds exactly to (31), and the solution has been observed to diverge from the equilibrium. In the tested applications, the algorithm converges rapidly if we select the adjustment speed parameter value $h^* = 0.3$.

Summary of the algorithm:

The initial values of the parameters are introduced in equation (33).

$$(m_1, m_2, a, r) = (m_1^0, m_2^0, a^0, r^0) \tag{33}$$

The values of (m_1, m_2, a, r) are sequentially updated. The iteration number is $i, i \in \{0, 1, ..., I\}$.

$$m_1^{i+1} = x_0 - m_2^i \tag{34}$$

$$m_2^{i+1} = x_T e^{-r^i T} - m_1^{i+1} e^{-2r^i T} \tag{35}$$

$$a^{i+1} = \frac{r^i (m_1^{i+1} - m_2^{i+1})}{y_0} \tag{36}$$

$$r^{i+1} = r^i + h\left(\frac{LN\left(\frac{\left(e^{-r^i T} \frac{r^i}{a^{i+1}} m_1^{i+1} - y_T \right)}{\left(\frac{r^i}{a^{i+1}} m_2^{i+1} \right)} \right)}{T} - r^i \right), \quad h = 0.3 \tag{37}$$

In case the solution does not converge in some other application, it is suggested that the adjustment speed parameter h is reduced to some value such that $0 < h < 0.3$.

The iteration algorithm in equations (33) to (37), is implemented in the computer code in the numerical appendix, and used to solve the coefficient estimation problems with empirical data.

Table 1.

The table shows the initial and terminal conditions and other parameters used in Case 0.

```
Initial and terminal conditions:

      x0 =  66150
      y0 =  18000
      xT =  52135
      yT =  200

  Other parameters:

      T =  30
      h =  .3

  Initial values of estimated parameters:

      a_0  =  .02
      b_0  =  .02
      r_0  =  .02
      m1_0 =  1
      m2_0 =  1
```

Table 2.

The table shows how the relative errors in the four equations develop, during the numerical iteration. The first row corresponds to iteration 1, and the final row corresponds to iteration 40. After 40 iterations, the absolute relative errors are less than 10^{-12} in all the four equations.

x0Err	y0Err	xTErr	yTErr
0.131331802328	0.406077556595	-0.066792339286	17.177549422846
0.046942030019	-0.044271399178	-0.001238733534	-2.424737248853
0.009794740047	-0.058589139025	-0.002141770808	-3.292782037948
0.002936436656	-0.035338780938	-0.000593458710	-1.960769298557
0.000903806604	-0.017852318448	-0.000083122461	-0.980041705441
0.000246411311	-0.008169564742	0.000011244902	-0.445823156844
0.000054828088	-0.003498047061	0.000014481506	-0.190341865254
0.000007624910	-0.001430777621	0.000007633629	-0.077753827990
-0.000001114673	-0.000567126973	0.000003306407	-0.030803137648
-0.000001551748	-0.000220022137	0.000001326350	-0.011947749717
-0.000000890008	-0.000084116219	0.000000513592	-0.004567327759
-0.000000414667	-0.000031836318	0.000000195333	-0.001728585582
-0.000000176158	-0.000011966158	0.000000073554	-0.000649706634
-0.000000071181	-0.000004476115	0.000000027533	-0.000243031058
-0.000000027914	-0.000001668771	0.000000010268	-0.000090605912
-0.000000010741	-0.000000620696	0.000000003819	-0.000033700634
-0.000000004082	-0.000000230489	0.000000001418	-0.000012514378
-0.000000001538	-0.000000085491	0.000000000526	-0.000004641745
-0.000000000577	-0.000000031684	0.000000000195	-0.000001720290
-0.000000000215	-0.000000011736	0.000000000072	-0.000000637198
-0.000000000080	-0.000000004345	0.000000000027	-0.000000235924
-0.000000000030	-0.000000001608	0.000000000010	-0.000000087327
-0.000000000011	-0.000000000595	0.000000000004	-0.000000032317
-0.000000000004	-0.000000000220	0.000000000001	-0.000000011958
-0.000000000002	-0.000000000081	0.000000000001	-0.000000004424
-0.000000000001	-0.000000000030	0.000000000000	-0.000000001637
-0.000000000000	-0.000000000011	0.000000000000	-0.000000000605
-0.000000000000	-0.000000000004	0.000000000000	-0.000000000224
-0.000000000000	-0.000000000002	0.000000000000	-0.000000000083
-0.000000000000	-0.000000000001	0.000000000000	-0.000000000031
-0.000000000000	-0.000000000000	0.000000000000	-0.000000000011
-0.000000000000	-0.000000000000	0.000000000000	-0.000000000004
-0.000000000000	-0.000000000000	0.000000000000	-0.000000000002
-0.000000000000	-0.000000000000	0.000000000000	-0.000000000001
0.000000000000	-0.000000000000	0.000000000000	-0.000000000000
0.000000000000	-0.000000000000	0.000000000000	-0.000000000000
0.000000000000	-0.000000000000	0.000000000000	-0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000

Table 3.

The table shows the estimated parameter values of Case 0.

a	=	5.347041320955464D−02
b	=	.0104491786465644
r	=	2.363729891362914D−02
m1	=	53434.08250957198
m2	=	12715.91749042803

Table 4.

The table shows the estimated force equations of Case 0.

$$x(t) = 53434.083 * EXP(-0.02363730 * t) + 12715.917 * EXP(0.02363730 * t)$$

$$y(t) = 23621.238 * EXP(-0.02363730 * t) - 5621.238 * EXP(0.02363730 * t)$$

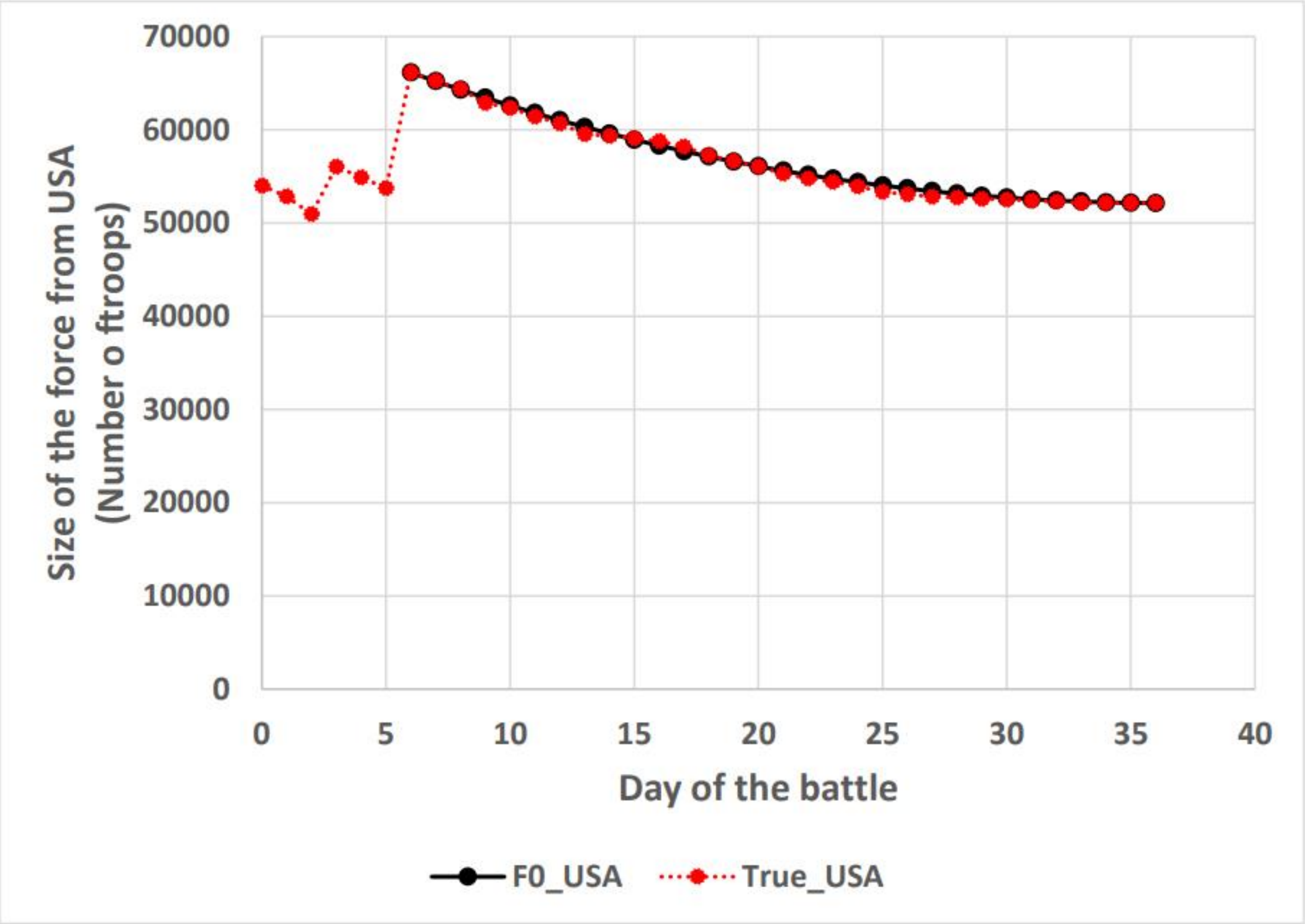


Figure 4.

Size of the force from USA, according to the true (empirical) time series and according to the model version F0_USA.

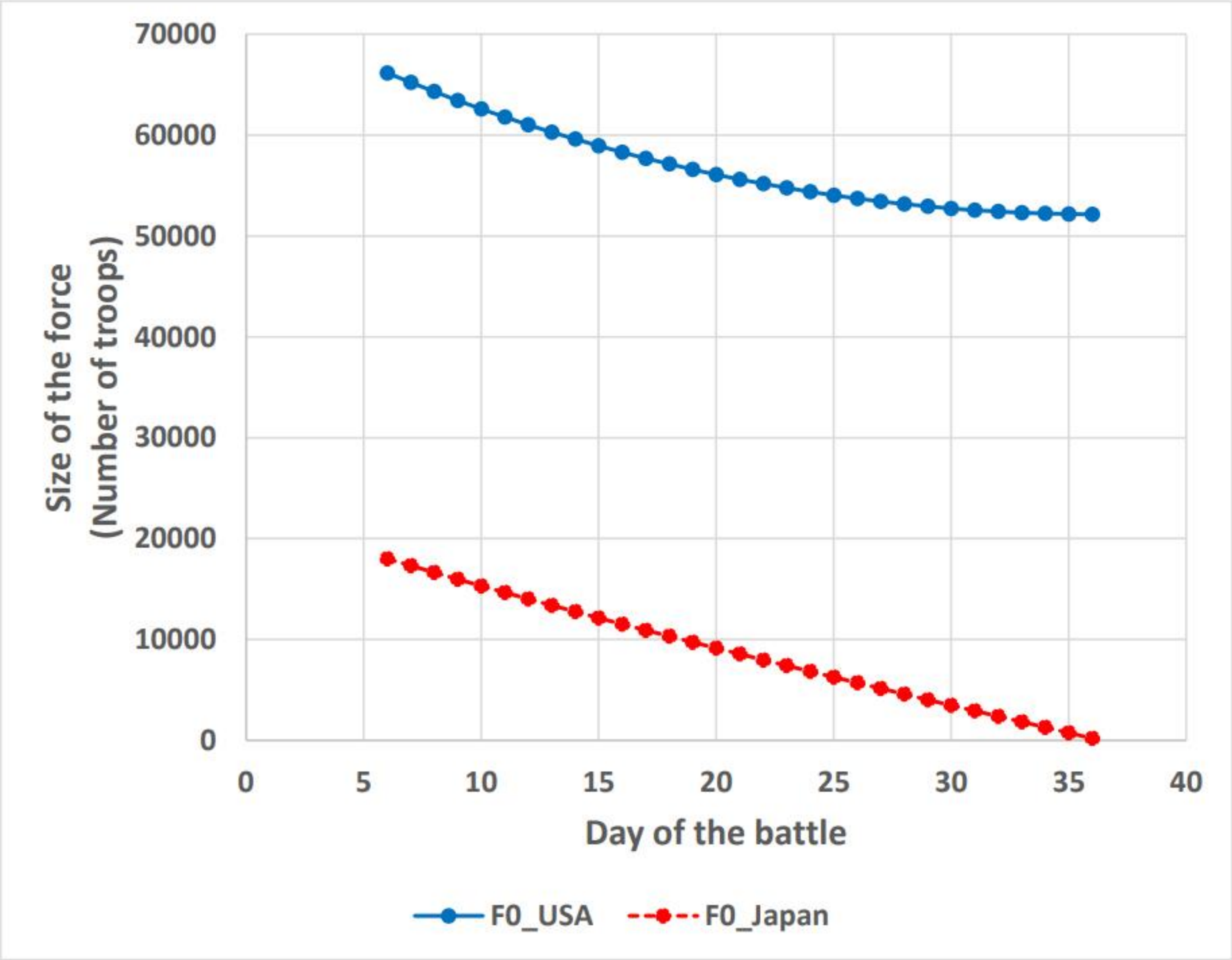


Figure 5.

Sizes of the forces from USA and Japan, according to the models F0_USA and F0_Japan.

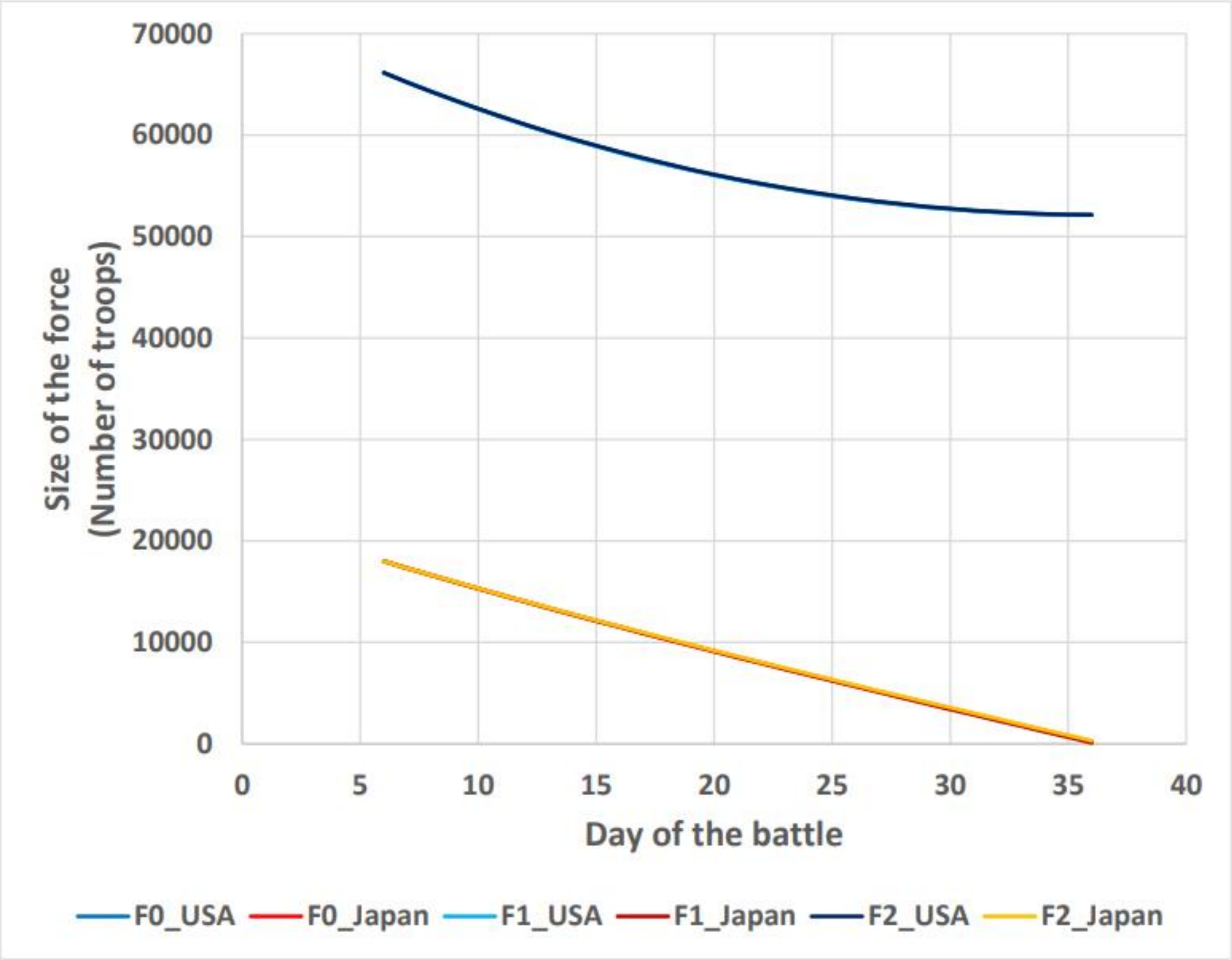


Figure 6.

Sizes of the forces from USA and from Japan, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T. In Case 0, Y_T =200, in Case 1, Y_T = 100, and in Case 2, Y_T =300. In Case 0, we have F0_USA and F0_Japan. In Case 1, we get F1_USA and F1_Japan. Case 2 gives F2_USA and F2_Japan.

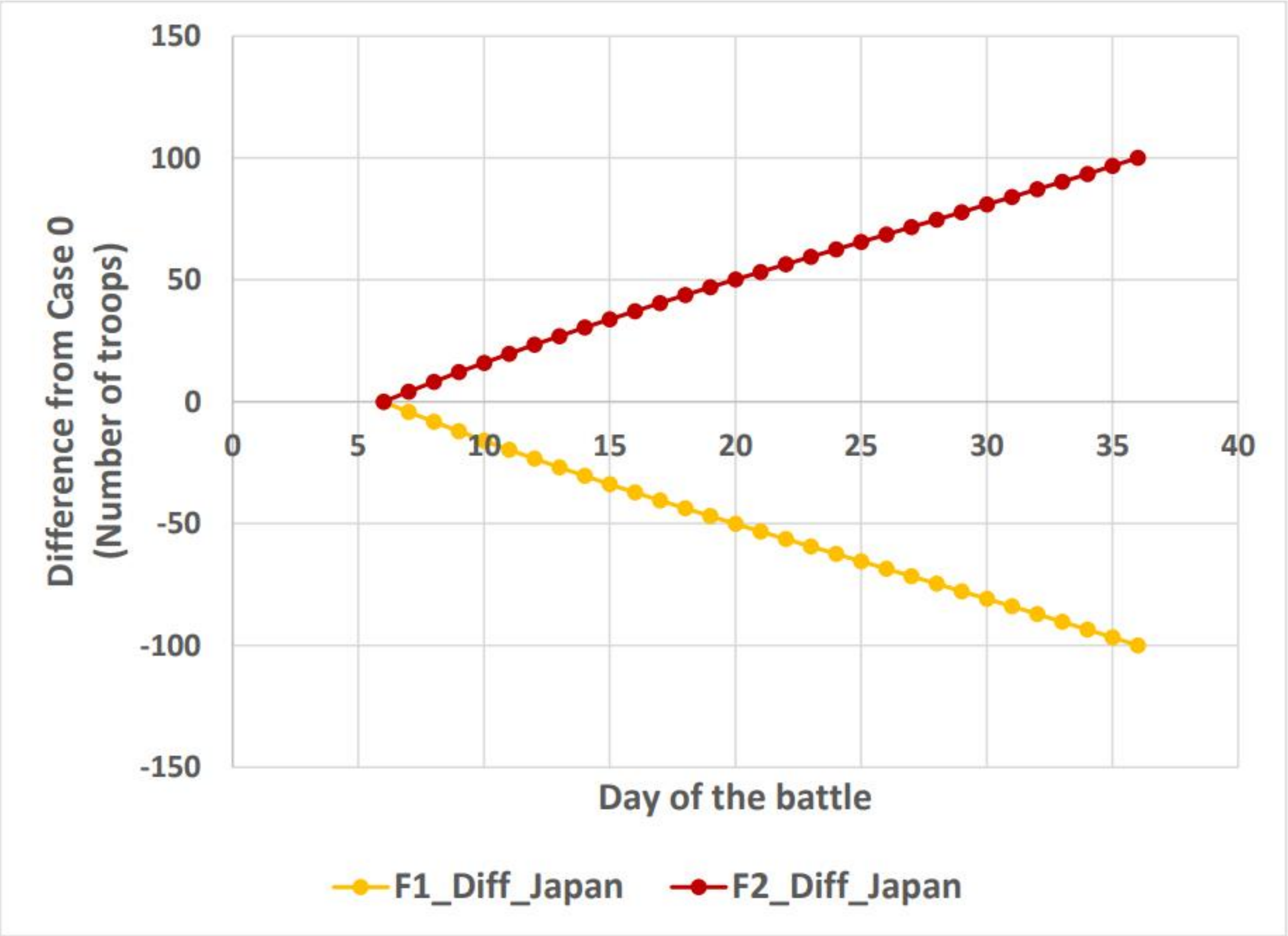


Figure 7.

Differences of the size of the force from Japan from Case 0, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T. In Case 0, Y_T =200, in Case 1, Y_T =100, and in Case 2, Y_T =300. In Case 0, we have F0_USA and F0_Japan. In Case 1, we get F1_USA and F1_Japan. Case 2 gives F2_USA and F2_Japan.

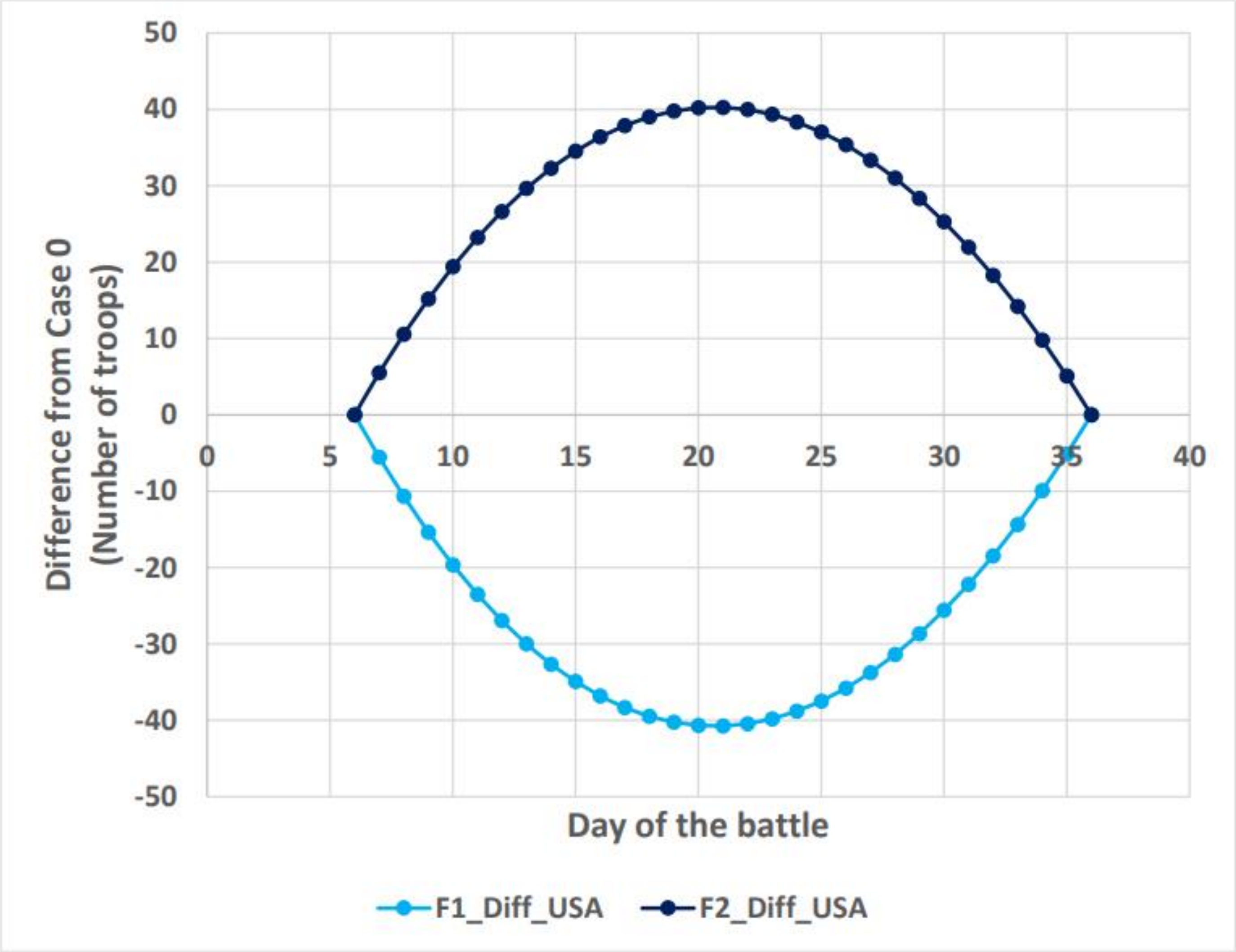


Figure 8.

Differences of the size of the force from USA from Case 0, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T. In Case 0, Y_T =200, in Case 1, Y_T =100, and in Case 2, Y_T =300. In Case 0, we have F0_USA and F0_Japan. In Case 1, we get F1_USA and F1_Japan. Case 2 gives F2_USA and F2_Japan.

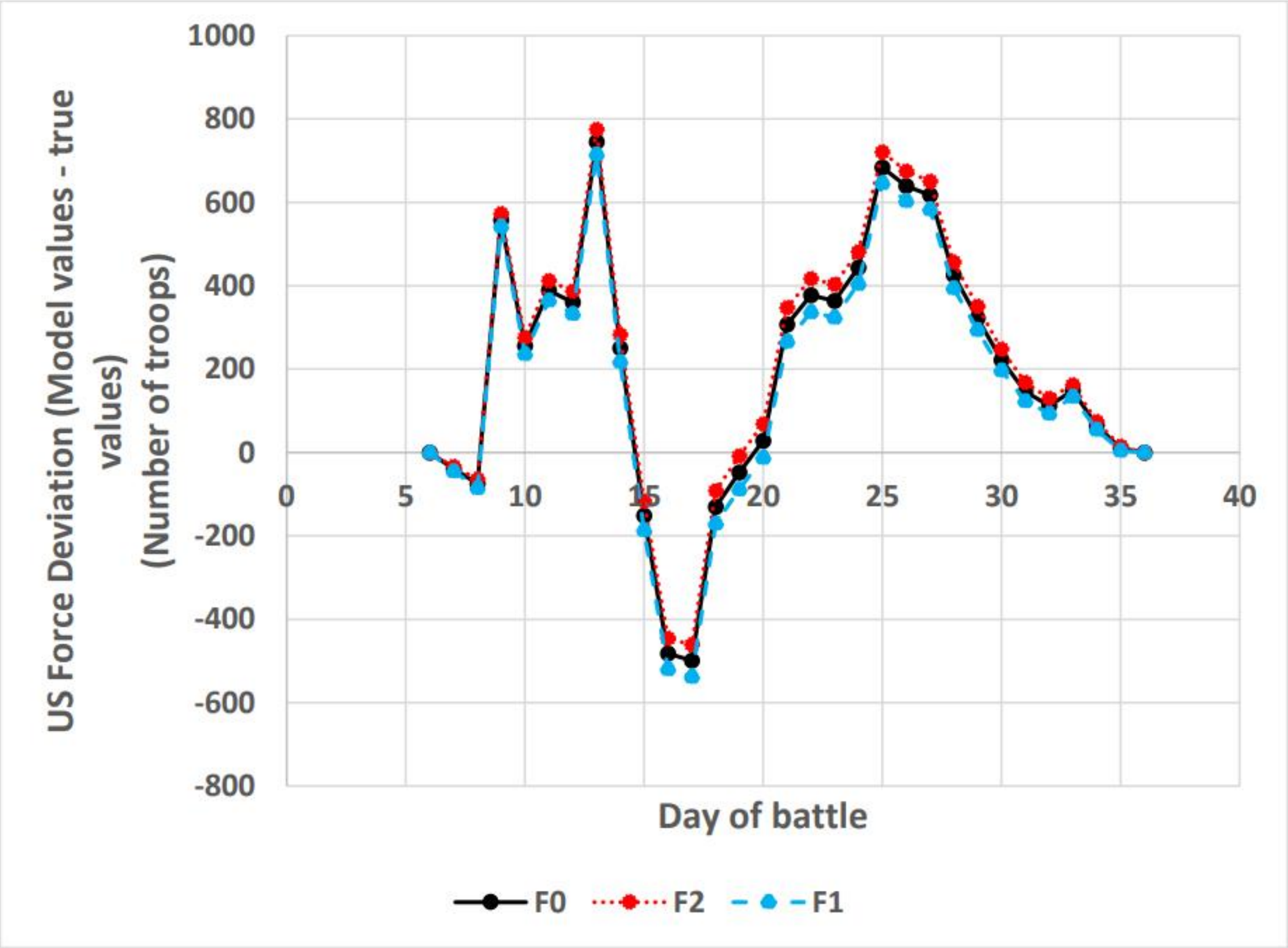


Figure 9.

Deviations of the size of the force from USA from the calculated values, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T. In Case 0, Y_T =200, in Case 1, Y_T =100, and in Case 2, Y_T =300. In Case 0, we have F0_USA and F0_Japan. In Case 1, we get F1_USA and F1_Japan. Case 2 gives F2_USA and F2_Japan.

Table 5.

Different estimations of the attrition coefficient values.

	Estimated value of a	**Estimated value of b**	**R2**
Lohmander Case 0	0.05347	0.01045	0.99997
Lohmander Case 1	0.05379	0.01051	0.99997
Lohmander Case 2	0.05315	0.01039	0.99997
Engel (1954)	0.0544	0.0106	0.9937
Braun (1993) via Engel (1954)	0.0544	0.0106	No information
Washburn and Kress (2009) via Engel (1954)	0.0544	0.0106	No information
Stymfal (2022)	0.0532	0.0105	0.9944

Parameter estimation based on discrete time and stochastic outcomes

$$a_t > 0 \wedge b_t > 0 \wedge x_t > 0 \wedge y_t > 0, \quad t \in \{0,1,...,T\} \tag{38}$$

$$a = \frac{a_0 + a_1}{2} \tag{39}$$

We can express the time dependent attrition coefficients as (40) and (41).

$$a_0 = a + s \tag{40}$$

$$a_1 = a - s \tag{41}$$

The coordinates at time t are $\left(x_t, y_t\right)$. These are recursively determined in (42) to (45).

$$\Delta x_1 = x_1 - x_0 = -a_0 y_0 \tag{42}$$

$$\Delta y_1 = y_1 - y_0 = -b_1 x_1 \tag{43}$$

$$\Delta x_2 = x_2 - x_1 = -a_1 y_1 \tag{44}$$

$$\Delta y_2 = y_2 - y_1 = -b_2 x_2 \tag{45}$$

The recursion (42) to (45) can be described as (46) to (49).

$$x_1 = x_0 - a_0 y_0 \tag{46}$$

$$y_1 = y_0 - b_1 x_1 \tag{47}$$

$$x_2 = x_1 - a_1 y_1 \tag{48}$$

$$y_2 = y_1 - b_2 x_2 \tag{49}$$

Now, we can determine how x_2 is affected by changing properties of the stochastic variable s .

From (46) and (48), we get (50). Via the earlier equations, (50) is further developed to (51).

$$x_2 = \left(x_0 - a_0 y_0\right) - a_1 y_1 \tag{50}$$

$$x_2 = \left(x_0 - \left(a + s\right) y_0\right) - \left(a - s\right)\left(y_0 - b_1\left(x_0 - a_0 y_0\right)\right) \tag{51}$$

$$x_2 = x_0 - \left(a + s\right) y_0 - \left(a - s\right) y_0 + \left(a - s\right) b_1\left(x_0 - \left(a + s\right) y_0\right) \tag{52}$$

$$x_2 = x_0 - 2a y_0 + \left(a - s\right) b_1 x_0 - \left(a - s\right)\left(a + s\right) b_1 y_0 \tag{53}$$

$$x_2 = \left(1 + \left(a - s\right) b_1\right) x_0 - \left(2a + \left(a^2 - s^2\right) b_1\right) y_0 \tag{54}$$

$$x_2 = \left(1 + \left(a - s\right) b_1\right) x_0 - \left(2a + \left(a^2 - s^2\right) b_1\right) y_0 \quad (54)$$

$$\frac{dx_2}{ds} = -b_1 x_0 + 2b_1 y_0 s \quad (55)$$

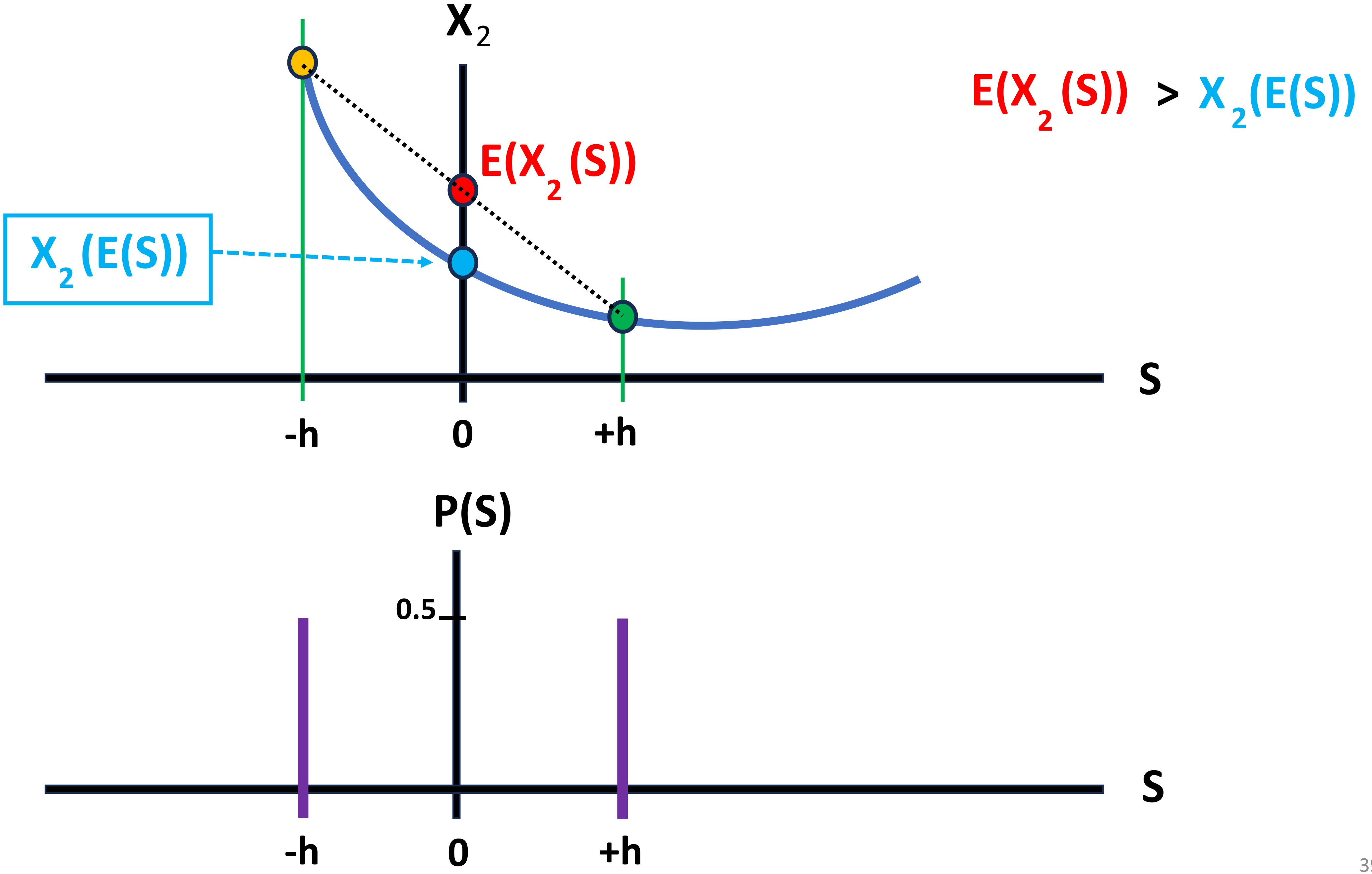
$$\frac{d^2 x_2}{ds^2} = 2b_1 y_0 > 0 \quad (56)$$

<u>Observation:</u>

x_2 may be regarded as a function of many parameters, including s. Compare equation (54). In equations (57) and (58), we simplify notation and write $x_2(s)$. According to equation (56), x_2 is a strictly convex function of s. From the Jensen's inequality (Jensen (1906)), we get the equations (57) and (58).

$$E\left(x_2\left(s\right)\right) > x_2\left(E\left(s\right)\right), \quad if\, \delta_s^{\,2} > 0 \quad (57)$$

$$E\left(x_2\left(s\right)\right) = x_2\left(E\left(s\right)\right), \quad if\, \delta_s^{\,2} = 0 \quad (58)$$



X_2
$E(X_2(S)) > X_2(E(S))$
$E(X_2(S))$
$X_2(E(S))$
S
-h
0
+h
P(S)
0.5
S
-h
0
+h

Terminal condition as expected value:

In a 2- period problem, we have the terminal condition found in equation (59). The expected value of x_2 is written as a function of (a, s), where s is a function of the standard deviation of s, δ_s.

$$E\left(x_2\left(a, s\left(\delta_s\right)\right)\right) = x_T \tag{59}$$

We are interested to see how the estimated a should be adjusted in case we know that δ_s increases, and we simultaneously want to make sure that the terminal condition (59) is satisfied.

Total differentiation gives equation (60). Clearly, as we see in equation (61), we cannot change the already known terminal value of the state variable.

$$\frac{dE\left(x_2\right)}{da}da + \frac{dE\left(x_2\right)}{d\delta_s}d\delta_s - dx_T = 0 \tag{60}$$

$$dx_T = 0 \tag{61}$$

Equations (60) and (61) lead to (62), which can be rewritten as (63).

$$\frac{dE(x_2)}{da}da+\frac{dE(x_2)}{d\delta_s}d\delta_s=0 \tag{62}$$

$$\frac{dE(x_2)}{da}da=-\frac{dE(x_2)}{d\delta_s}d\delta_s \tag{63}$$

The derivative of the parameter a, the estimated expected attrition coefficient, with respect to the standard deviation of the attrition coefficient, δ_s, is found in equation (64).

$$\frac{da}{d\delta_s}=\frac{-\left(\frac{dE(x_2)}{d\delta_s}\right)}{\left(\frac{dE(x_2)}{da}\right)} \tag{64}$$

In order to determine the sign of the derivative in equation (64), must know the sign of the derivative of x_2 with respect to a, which is found in (65). Equation (65) can be reformulated to (66) and (67).

$$\frac{dx_2}{da} = -2y_0 + b_1 x_0 - 2b_1 y_0 a \tag{65}$$

$$\frac{dx_2}{da} = -2y_0 \left(1 + b_1 a\right) + b_1 x_0 \tag{66}$$

$$\frac{dx_2}{da} = b_1 x_0 \left(-2 \frac{\left(1 + b_1 a\right)}{b_1} \frac{y_0}{x_0} + 1 \right) \tag{67}$$

Equation (68) shows a combination of three different assumptions, which makes sure that the sign of the derivative of x_2 with respect to a is strictly negative. The first listed assumption follows from the earlier assumptions in this paper. The second assumption is satisfied in case $b_1 < 0.1$, which is normal in most battles. (Compare the attrition coefficient values is Table 5.) The third assumption is a constraint on the ratio between the sizes of the initial forces: The initial size of force Y is at least 5% of the initial size of force X. That assumption is probably relevant in almost all real battles. Compare the initial force sizes reported in Table 1. In case the assumptions in (68) are true, then equation (69) follows.

$$\left(b_1 x_0 > 0\right) \wedge \left(\frac{\left(1 + b_1 a\right)}{b_1} > 10 \right) \wedge \left(\frac{1}{20} \le \frac{y_0}{x_0} \right) \Rightarrow \frac{dx_2}{da} < 0 \tag{68}$$

$$\left(\frac{dE\left(x_2\right)}{d\delta_s} > 0 \wedge \frac{dE\left(x_2\right)}{da} < 0 \right) \Rightarrow \frac{da}{d\delta_s} = \frac{-\left(\frac{dE\left(x_2\right)}{d\delta_s} \right)}{\left(\frac{dE\left(x_2\right)}{da} \right)} > 0 \tag{69}$$

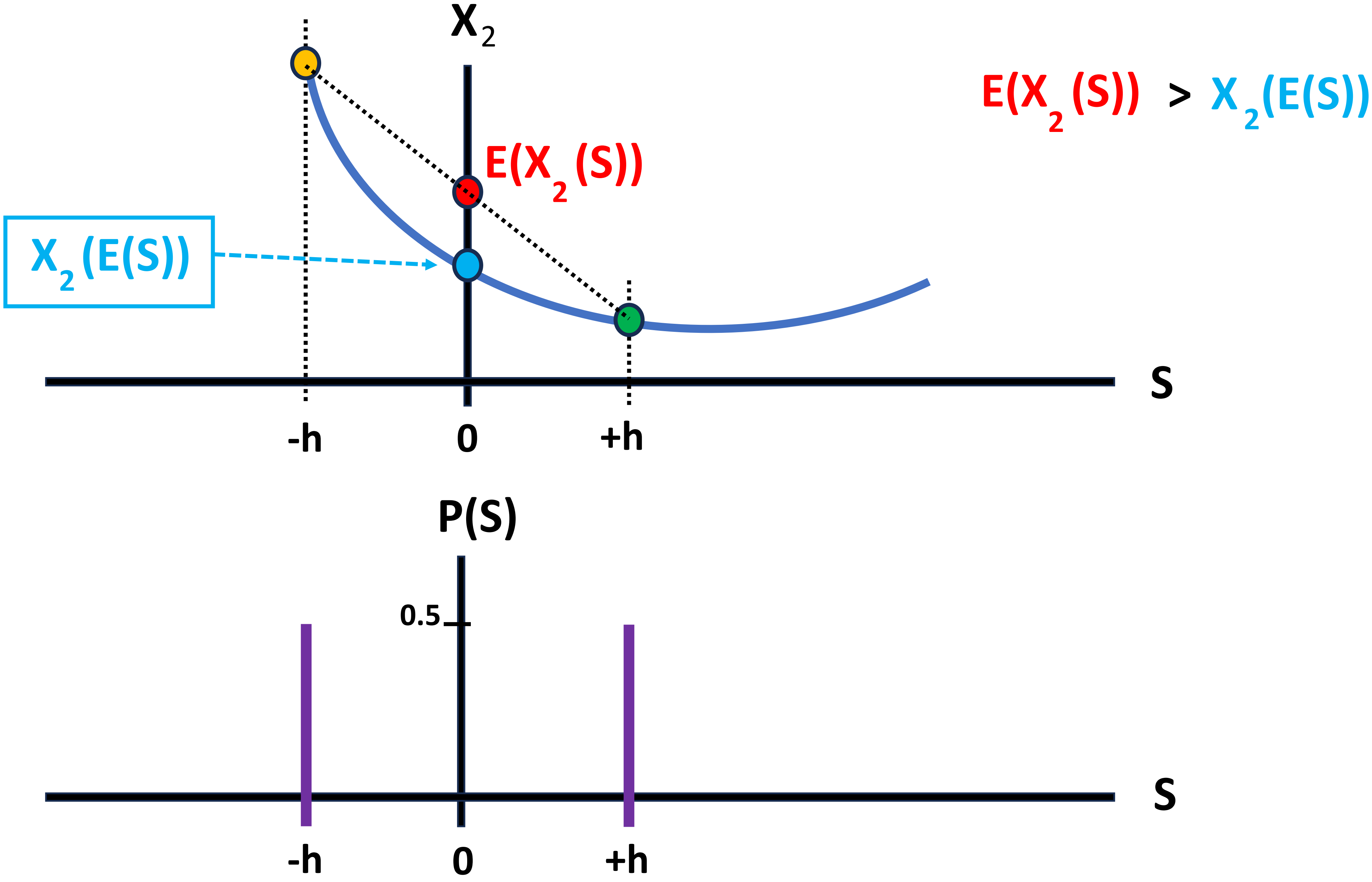
$$\left(\frac{dE(x_2)}{d\delta_s} > 0 \wedge \frac{dE(x_2)}{da} < 0\right) \Rightarrow \frac{da}{d\delta_s} = \frac{-\left(\frac{dE(x_2)}{d\delta_s}\right)}{\left(\frac{dE(x_2)}{da}\right)} > 0 \tag{69}$$

Some interpretations of equation (69) and the earlier assumptions are the following: We are interested to see how the estimated expected attrition coefficient a should be adjusted in case δ_s changes. Simultaneously, we want to make sure that the terminal condition (59) is satisfied. The estimated expected attrition coefficient a is a strictly increasing function of δ_s. In other words; If the attrition coefficients contain more stochastic variation, and the terminal size of force X is constant, then the estimated value of the expected attrition coefficient increases.

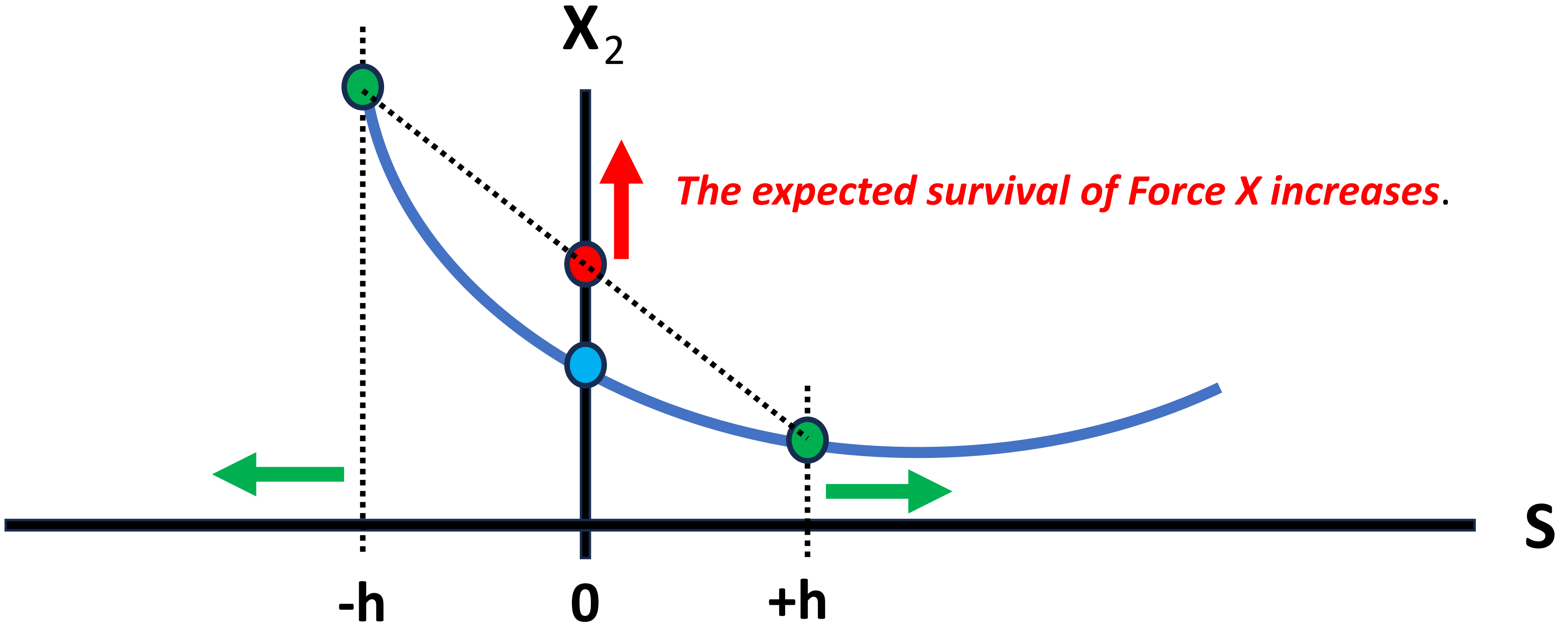
Generalization:

In case the reader prefers a more general version of the theory developed in (68) and (69), we may study equation (70). Equations (64) and (67) imply equation (70). The first assumption written in equation (70) follows from the earlier assumptions in this paper. The second assumption is that the ratio $\frac{y_0}{x_0}$ exceeds a particular value, determined by the parameters (a,b_1). In case $b_1 = 0.1$, $\frac{y_0}{x_0}$ should exceed 0.05, to satisfy the constraint, for all $a \geq 0$. In case $b_1 = 0.2$, $\frac{y_0}{x_0}$ should exceed 0.10, to satisfy the constraint, for all $a \geq 0$. If the constraint on the initial force ratios is satisfied, then equation (69) is satisfied, which is also clear from equation (70). In other words; If the attrition coefficients contain more stochastic variation, and the terminal size of force X is constant, then the estimated value of the expected attrition coefficient increases.

$$\left(b_1 x_0 > 0\right) \wedge \left(\frac{y_0}{x_0} > \frac{b_1}{2\left(1+b_1 a\right)}\right) \Rightarrow \left(\frac{dx_2}{da} < 0\right) \Rightarrow \left(\frac{da}{d\delta_s} = \frac{-\left(\frac{dE\left(x_2\right)}{d\delta_s}\right)}{\left(\frac{dE\left(x_2\right)}{da}\right)} > 0\right) \tag{70}$$



X2
E(X2(S)) > X2(E(S))
E(X2(S))
X2(E(S))
S
-h
0
+h
P(S)
0.5
S
-h
0
+h



The variation in the efficiency of Force Y, ”the attrition parameter a”, increases (from the assumed value 0).

The expected survival of Force X increases. (This means that the expected attrition by Force Y is reduced.)

However, the ***terminal value of the size of Force X is an empirical fact*** and does not change.

Hence, it is necessary that the **expected efficiency of Force Y, ”the attrition parameter a”, increases**.

In other words: ***The true value of the expected attrition parameter ”a” has to increase.***

Conclusions:

This study has shown the following:

It is possible to determine the expected attrition coefficients of a battle, if the initial and terminal sizes of the forces of the involved parties are known, and the general solution to the relevant differential equation system can be derived.

This means that detailed statistical data tables, representing the time series of the sizes of the involved forces, are not necessary.

This is an important conclusion since it is often very difficult, costly, dangerous and/or impossible to get access to detailed and reliable military statistical data, particularly during wars that have not yet ended.

In the earlier mentioned articles on the battle of Iwo Jima, the authors of those articles used different statistical procedures and approximations to estimate the attrition coefficients.

Now, with the new estimation procedure, based on a general differential equation system solution and a numerical iteration algorithm, it is possible to rapidly obtain almost identical estimates of the attrition parameters.

Furthermore, with the new procedure, it is also possible to instantly, in less than a second, determine how possible changes of different parameters, such as the not exactly known terminal size of the Japanese force, influence the estimated attrition parameters.

The new procedure automatically reports not only the estimated attrition coefficients, but also the equations that describe the dynamics of the involved forces, as explicit functions of time.

The expected size of Force X at a later point in time is a strictly increasing function of the risk in the attrition coefficients of Force Y, during earlier points in time.

If the attrition coefficients of Force Y are stochastic, and the expected attrition coefficients are estimated via regression analysis based on the complete detailed time series of the involved forces, then the estimated expected attrition coefficients should be larger, than if the attrition coefficient estimates are calculated based on the assumption that the attrition coefficients never change.

Thank you for your time!

Thank you also for a very nice conference!

Peter Lohmander

https://www.lohmander.com/Information/Ref.htm

Attrition coefficient estimations

via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions

by
Peter Lohmander

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Prof Dr

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